THE CIRCLE

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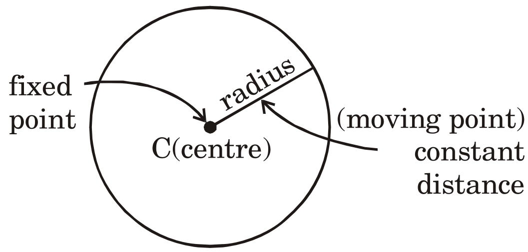
A circle is the locus of a point which moves, so that its distance from a fixed point, called the centre, is equal to a given distance.

The fixed point is called the centre of the circle and the constant distance is called its radius.

i.e., CP = constant distance = Radius



1. Standard equation of a circle

If the centre of circle is at origin and radius is a, then the equation of circle is given by

$$\boxed{x^2 + y^2 = a^2}$$

and this is also known as standard equation of circle.

In case the centre is a point (h, k), and not origin then the equation of circle is given by

$$\boxed{(x - h)^2 + (y - k)^2 = a^2}$$

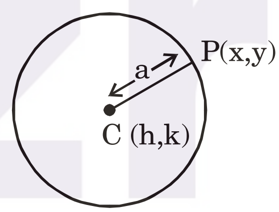
CP = radius = a

according to constraint defined by circle

distance CP = radius = a

$\Rightarrow \quad (x - h)^2 + (y - k)^2 = a^2$

This form is known as central form of equation of circle.



2. General Equation of a Circle :

General equation of second degree in x and y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

The equation of a circle with centre (α, β) and radius r is

$$(x - \alpha)^2 + (y - \beta)^2 = r^2 \quad ...(ii)$$

$$x^2 + y^2 - 2\alpha x + 2\beta y + (\alpha^2 + \beta^2 - r^2) = 0$$

Comparing (i) and (ii)

$$\frac{1}{a} = \frac{1}{b} = \frac{0}{h} = \frac{g}{-\alpha} = \frac{f}{-\beta} = \frac{c}{\left(\alpha^2 + \beta^2 - r^2\right)}$$

$\Rightarrow$ $a = b$ **i.e. coefficient of x^2 = coefficient of y^2**

$h = 0$ **i.e., coefficient of $xy = 0$**

The general equation of a circle is

$$\boxed{x^2 + y^2 + 2gx + 2fy + c = 0} \qquad ...(iii)$$

where g, f and c are constants

To find the centre and radius.

Equation (i) can be written as

$$(x+g)^2 + (y+f)^2 = \left[\sqrt{\left(g^2 + f^2 - c\right)}\right]^2$$

Comparing with the equation of the circle given in (ii), h = – g, k = – f

and $a = \sqrt{(g^2 + f^2 - c)}$

∴ Coordinates of the centre are **(– g, – f)** and radius = $\sqrt{(g^2 + f^2 - c)}$

Important remarks

(i) If $g^2 + f^2 - c > 0$, equation (i) represents real circle with centre (– g, – f).

(ii) If $g^2 + f^2 - c = 0$, the equation (i) represents a circle whose centre is (– g, – f) and radius is zero i.e., the circle coincides with the centre and so it represents a point (– g, – f). It is therefore, called a point circle.

(iii) If $g^2 + f^2 - c < 0$, radius of the circle is imaginary. In this case, there are no real points on the circle and so it is called a virtual circle or imaginary circle.

(iv) Dependence of the circle on three unknown parameters : The equation (i) i.e., $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three unknown quantities g, f, c. Hence for determining the equation of a circle, three conditions are required.

Note : In case the equation of circle is given by $ax^2 + ay^2 + 2gx + 2fy + c = 0$. Divide the equation by a first. i.e. $x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + c = 0$

then centre is $\left(\frac{-g}{a}, \frac{-f}{a}\right)$ & radius $\left(\sqrt{\frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a}}\right.$

Remember that for finding centre and radius of circle the coefficients of x & y should be equal to 1.

Note : Rule for finding the centre and radius of a circle :

(i) Make the coefficients of x^2 and y^2 equal to 1 and right hand side equal to zero.

(ii) Then co-ordinates of centre will be (α, β),

where $\alpha = -\frac{1}{2}(\text{coefficient of } x)$ and $\beta = -\frac{1}{2}(\text{coefficient of } y)$

(iii) Radius = $\sqrt{\alpha^2 + \beta^2 - (\text{constant term})}$

3. Other Forms of Circle

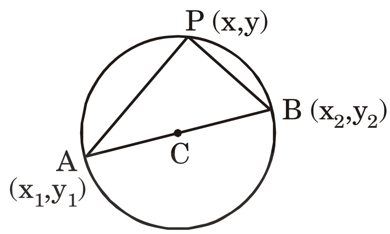
3.1 Diametric Form

The equation to the circle which is described on the line joining the points (x_1, y_1) and (x_2, y_2) as diameter.

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Proof :

Let A be the point (x_1, y_1) and B the point (x_2, y_2), and let the coordinates of any point P on the circle be h and k. (Where AB is a diameter)



The equation to AP is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1) \qquad ...(1)$$

and the equation to BP is

$$y - y_2 = \frac{k - y_2}{h - x_2}(x - x_2) \qquad ...(2)$$

But, since APB is a semicircle, the angle APB is a right angle, and hence the straight lines (1) and (2) are at right angles.

Hence, we have

$$\frac{k - y_1}{h - x_1} \cdot \frac{k - y_2}{h - x_2} = -1$$

i.e., $$(h - x_1)(h - x_2) + (k - y_1)(k - y_2) = 0$$

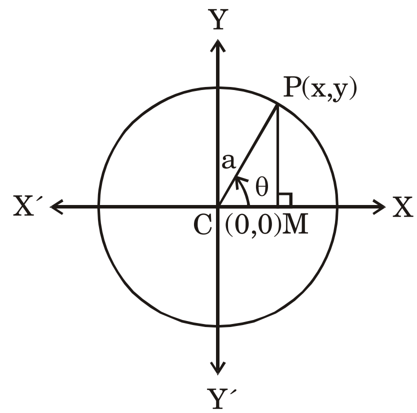
But this is the condition that the point (h, k) may lie on the curve whose equation is,

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

This therefore, is the required equation.

3.2. Parametric form :

If the radius of a circle whose centre is at C (0, 0) makes an angle θ with the positive direction of x-axis, then θ is called the parameter.



Let CP = a

$\therefore$ CM = x, PM = y

$\Rightarrow$ x = a cos θ, y = a sin θ

Hence **(a cos θ, a sin θ)** or 'θ' are the **parametric co-ordinates** of the circle $x^2 + y^2 = a^2$ and x = a cos θ and y = a sin θ are called parametric equations of the circle $x^2 + y^2 = a^2$ with parameters a and θ. (θ ≤ < θ 2π)

Note :

1. The parametric co-ordinates of any point on the circle

$$(x - h)^2 + (y - k)^2 = a^2$$

are given by (h + a cos θ, k + a sin θ) (0 ≤ θ < 2π)

and parametric equations of the circle

$(x - h)^2 + (y - k)^2 = a^2$ are x **=** h **+** a **cos θ,** y **=** k **+** a **sin θ.**

2. Equation of the chord of the circle $x^2 + y^2 = a^2$ joining (a cos α, a sin α) and (a cos β, a sin β) is

$$x \cos\left(\frac{\alpha+\beta}{2}\right) + y \sin\left(\frac{\alpha+\beta}{2}\right) = a \cos\left(\frac{\alpha-\beta}{2}\right)$$

3.3. Equation of circle through 3 points

If three points A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) are non-collinear, an unique circle passes through A, B and C. To find the equation of this circle, we can proceed in the following way :

Let required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

Since this circle passes through A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3), we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad ...(ii)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad ...(iii)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \quad ...(iv)$$

Solving these three, we can find g, f, c and hence the equation of the circle.

Students acquainted with determinants, can put the equation of this circle in determinant form as follows :

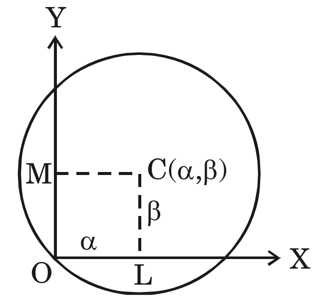
Eliminating g, f and c from (i) and (iv), we have

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0 \quad ...(v)$$

(vi) is an equation of the circle passing through the points A, B and C.

4. EQUATION OF A CIRCLE IN SPECIAL CASES

(i) When the circle passes through the origin **:**



Let the equation of circle be

$(x - \alpha)^2 + (y - \beta)^2 = a^2$

Since it passes through O (0, 0)

$\therefore \quad (0 - \alpha)^2 + (0 - \beta)^2 = a^2$

or $\quad a^2 = \alpha^2 + \beta^2$

Hence the equation of the circle in this case will be

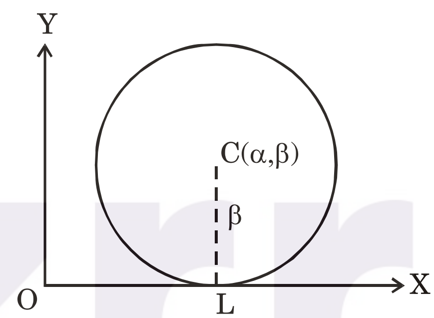
$$(x - \alpha)^2 + (y - \beta)^2 = \alpha^2 + \beta^2$$

i.e. $\quad x^2 + y^2 - 2\alpha x - 2\beta x = 0$

Note : Thus if a circle passes through the origin its equation may be taken as

$$\boxed{x^2 + y^2 + 2gx + 2fy = 0}$$

(ii) When the circle touches x-axis



Let the centre of the circle be C (α, β)

Since the circle touches the x-axis the radius of the circle $= CL = |\beta|$

Hence the equation of the circle is

$$(x - \alpha)^2 + (y - \beta)^2 = |\beta|^2$$

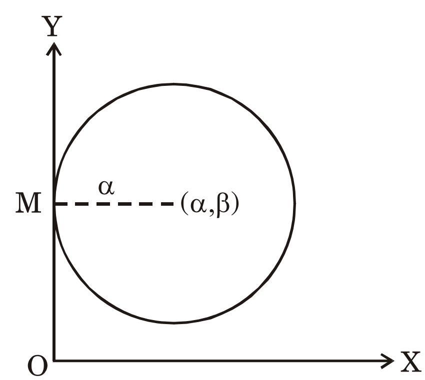
or $\quad (x - \alpha)^2 + (y - \beta)^2 = \beta^2$

Note : If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches x-axis then $g^2 + f^2 - c = f^2$. i.e. $c = g^2$

Hence the equation of a circle touching x-axis may be taken as

$$\boxed{x^2 + y^2 + 2gx + 2fy + g^2 = 0}$$

(iii) When the circle touches y-axis.



Let the centre of the circle be C (α, β)

Since the circle touches the y-axis; the radius of the circle $= CM = |\alpha|$

Hence the equation of the circle is

$$(x - \alpha)^2 + (y - \beta)^2 = |\alpha|^2$$

or $\quad (x - \alpha)^2 + (y - \beta)^2 = \alpha^2$

Note : The equation of a circle touching y-axis may be taken

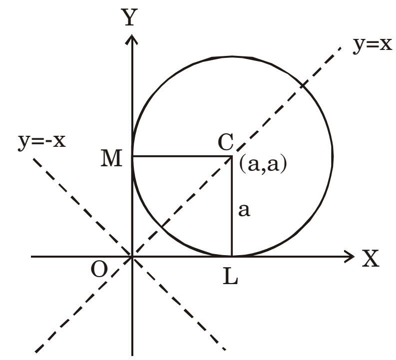
as $\quad \boxed{x^2 + y^2 + 2gx + 2fy + f^2 = 0}$

(iv) <u>When the circle touches both the axes.</u>

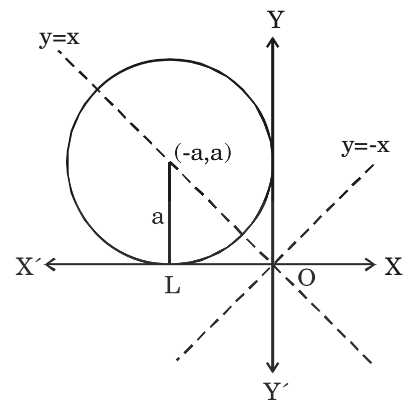
In this case $|\alpha| = |\beta| = a$.

Hence the equation of the circle, in the first quadrant, is

$$\boxed{(x - a)^2 + (y - a)^2 = a^2} \qquad ...(1)$$



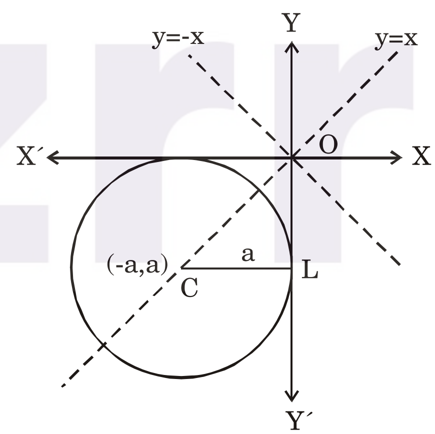
The equation of the circle in the second quadrant is



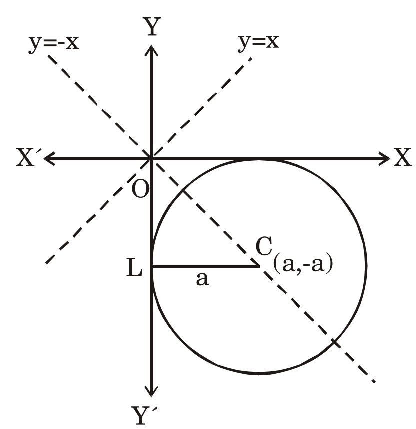
$$\boxed{(x + a)^2 + (y - a)^2 = a^2} \qquad ...(2)$$

The equation of the circle in the third quadrant is

$$\boxed{(x + a)^2 + (y + a)^2 = a^2} \qquad ...(3)$$



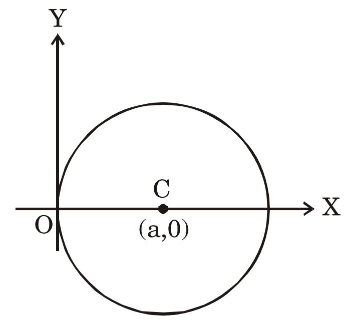
The equation of the circle in the fourth quadrant is



$$\boxed{(x - a)^2 + (y + a)^2 = a^2} \qquad ...(4)$$

Thus if a circle touches both axes its centre may be taken as $(\pm a, a)$ where a is positive or negative and radius as $|a|$.

(v) <u>When the circle passes through the origin and centre lies on x-axis :</u>



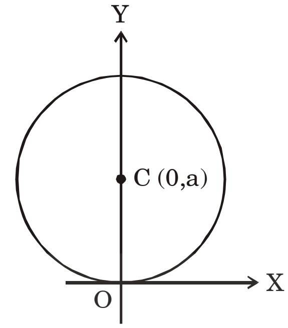
Let centre of circle be C (a, 0)

$\therefore$ radius = a

$\therefore$ Equation of circle is $(x - a)^2 + (y - 0)^2 = a^2$

or $\boxed{x^2 + y^2 - 2ax = 0}$

(vi) <u>When the circle passes through the origin and centre lies on y-axis :</u>



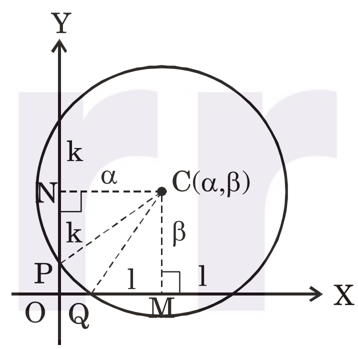
Let centre of circle be C (0, a)

$\therefore$ radius = a

$\therefore$ Equation of circle is $(x - 0)^2 + (y - a)^2 = a^2$

or $\boxed{x^2 + y^2 - 2ay = 0}$

(vii) <u>When the circle cut off intercepts on x-axis and y-axis of lengths 2l and 2k and not passing through origin :</u>



Let centre be (α, β)

$\therefore$ radius = CP = CQ = λ (say)

$(CP)^2 = (CQ)^2 = \lambda^2$

$\alpha^2 + k^2 = \beta^2 + l^2 = \lambda^2$

$\therefore$ $\alpha = \sqrt{\lambda^2 - k^2}$ [for I quadrant]

and $\beta = \sqrt{\lambda^2 - l^2}$

$\therefore$ Equation of circle is $(x - \sqrt{\lambda^2 - k^2})^2 + (y - \sqrt{\lambda^2 - l^2})^2 = \lambda^2$

Illustration 1

For what value(s) of λ, the equation $(10 - \lambda^2)x^2 + (\lambda^2 - 8)y^2 + (3 - \lambda)yx - 10x + 4y + 3 = 0$ represent a real circle.

Solution :

For equation of circle we must satisty the following 2 conditions.

Condition 1 : coeff. of x^2 = coeff. of y^2

$\Rightarrow$ $10 - \lambda^2 = \lambda^2 - 8$ $\Rightarrow$ $\lambda^2 = 9 \Rightarrow$ $\lambda = \pm 3$

Condition 2 : and coeff. of xy = 0

$\Rightarrow$ $3 - \lambda = 0$ $\Rightarrow$ $\lambda = 3$

So, the common value is $\lambda = 3$.

Illustration 2

Find the equation of the circle which touches the axes and whose centre lies on the line $x - 2y = 3$.

Solution :

Since the circle touches both the axes, therefore its centre will be $(a, \pm a)$ and radius will be $|a|$, where a is a positive or negative number.

Given line is $x - 2y = 3$ ...(i)

Case I : When centre is $(a, \pm a)$

Since $(a, \pm a)$ lies on line (i)

$\therefore$ centre of the circle is $(-3, -3)$ and radius $= |-3| = 3$.

Hence equation of the circle will be

$(x + 3)^2 + (y + 3)^2 = 3^2$ or $x^2 + y^2 + 6x + 6y + 9 = 0$

Case II : When centre is $(a, -a)$

Since $(a, -a)$ lies on line (i)

$\therefore$ $a + 2a = 3$

$\therefore$ $a = 1$

$\therefore$ centre of the circle is $(1, -1)$ and radius $= |1| = 1$

Hence equation of the circle will be

$(x - 1)^2 + (y + 1)^2 = 1^2$ or $x^2 + y^2 - 2x + 2y + 1 = 0$

Illustration 3

Find the equation of the circle passing through the origin and cuts off chords of length 4 and 6 on the positive side of x and y-axis respectively.

Solution :

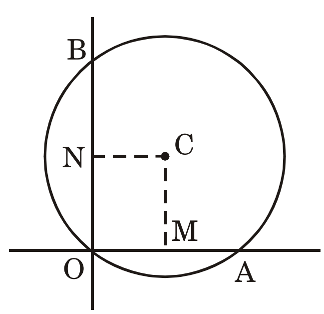
If the centre C be (h, k) then the values of h, k will be length of perpendicular CM and CN from C to x- and y-axis respectively.

$\Rightarrow$ M and N are mid points of chords OA and OB

$\Rightarrow$ OM = 2, ON = 3

$\Rightarrow$ C is (2, 3)

Also radius OC $= \sqrt{2^2 + 3^2} = \sqrt{13}$



Hence the equation of the required circle is $(x - 2)^2 + (y - 3)^2 = \left(\sqrt{13}\right)^2$

Illustration 4

Find the equation to the circle which passes through the points (1, 0), (0, 6) and (3, 4).

Solution :

Let the equation to the circle be,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(1)$$

Since, the three points, whose coordinates are given, satisfy this equation, we have

$$1 + 2g + c = 0 \quad ...(2)$$

$$36 + 12f + c = 0 \quad ...(3)$$

and $$25 + 6g + 8f + c = 0 \quad ...(4)$$

Subtracting (2) from (3) and (3) from (4), we have

$$2g + 12f = 35$$

and $$6g + 20f = 11$$

Hence, $f = \dfrac{47}{8}$ and $g = -\dfrac{71}{4}$

Equation (2), then gives $c = \dfrac{69}{2}$

Substituting these values in (1) the required equation is,

$$4x^2 + 4y^2 - 142x + 47y + 138 = 0$$

Illustration 5

Find the equation to the circle which touches the axis of y at a distance + 4 from the origin and cuts off an intercept 6 from the axis of x.

Solution :

Any circle is, $x^2 + y^2 + 2gx + 2fy + c = 0$

This meets the axis of y in points given by

$$y^2 + 2fy + c = 0$$

The roots of this equation must be equal and each equal to 4, so that it must be equivalent to

$$(y - 4)^2 = 0$$

Hence, $2f = -8$ and $c = 16$

The equation to the circle is then

$$x^2 + y^2 + 2gx - 8y + 16 = 0$$

This meets the axis of x in points given by,

$$x^2 + 2gx - 8y + 16 = 0$$

i.e., at points distant

$-g+\sqrt{g^2-16}$ and $-g-\sqrt{g^2-16}$

Hence, $$6 = 2\sqrt{g^2-16}$$

Therefore, $g = \pm 5$, and the required equation is, $x^2 + y^2 \pm 10x - 8y + 16 = 0$

Therefore, are therefore, two circle satisfying the given conditions. This is geometrically obvious.

Illustration 6

The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and the ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter.

Solution :

Given equations are

$$x^2 + 2ax - b^2 = 0 \quad \text{...(i)}$$

and $$x^2 + 2px - q^2 = 0 \quad \text{...(ii)}$$

Let the roots of equation (i) be α and β and and those of equation (ii) be γ and δ, then

$$\left.\begin{array}{r}\alpha+\beta=-2a\\ \alpha\beta=-b^2\end{array}\right\} \text{ and } \begin{array}{l}\gamma+\delta=-2p\\ \gamma\delta=-q^2\end{array}$$

Let $A = (\alpha, \gamma)$ and $B = (\beta, \delta)$

Now equation of the circle whose diameter is AB will be

$(x - \alpha)(x - \beta) + (y - \gamma)(y - \delta) = 0$

or $x^2 + y^2 - (\alpha + \beta)x - (\gamma + \delta)y + \alpha\beta + \gamma\delta = 0$

or $x^2 + y^2 + 2ax + 2py - (b^2 + q^2) = 0$ ...(iii)

Centre of circle (iii) is $(-a, -p)$

and radius = $\sqrt{(-a)^2 + (-p)^2 + b^2 + q^2} = \sqrt{a^2 + b^2 + p^2 + q^2}$

Illustraion 7

If the parametric of form of a circle is given by

(i) $x = -4 + 5\cos\theta$ **and** $y = -3 + 5\sin\theta$

(ii) $x = a\cos\alpha + b\sin\alpha$ **and** $y = a\sin\alpha - b\cos\alpha$

find its cartesian form.

A

Solution :

(i) The given equations are

$$x = -4 + 5\cos\theta \text{ and } y = -3 + 5\sin\theta$$

or $(x + 4) = 5\cos\theta$ ...(1)

and $(y + 3) = 5\sin\theta$ ...(2)

Squaring and adding (1) and (2), then

$$(x + 4)^2 + (y + 3)^2 = 5^2$$

or $$(y + 4)^2 + (y + 3)^2 = 25$$

(ii) The given equations are

$$x = a \cos \alpha + b \sin \alpha \quad ...(1)$$

$$y = a \sin \alpha - b \cos \alpha \quad ...(2)$$

Squaring and adding (1) and (2), then

$$x^2 + y^2 = (a \cos \alpha + b \sin \alpha)^2 + (a \sin \alpha - b \cos \alpha)^2$$

$$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$$

Illustration 8

Show that the four points (1, 0), (2, – 7), (8, 1) and (9, – 6) are concyclic.

Solution :

Since the given four points are concyclic, we are to show that they lie on a circle. Let the general equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(1)$$

has three parameters, it is sufficient to obtain the equation of the circle passing through any three of these points. For concyclic, the fourth point should lie on this circle.

Let three points A (1, 0), B (2, – 7) and D (8, 1) lie on (1) then

$$1 + 0 + 2g + 0 + c = 0 \text{ or } 1 + 2g + c = 0 \quad ...(2)$$

$$(2)^2 + (-7)^2 + 2g(2) + 2f(-7) + c = 0$$

or $$53 + 4g - 14f + c = 0 \quad ...(3)$$

and $$(8)^2 + (1)^2 + 2g(8) + 2f(1) + c = 0$$

$$\Rightarrow \quad 65 + 16g + 2f + c = 0 \quad ...(4)$$

Now subtracting (2) from (3), we get

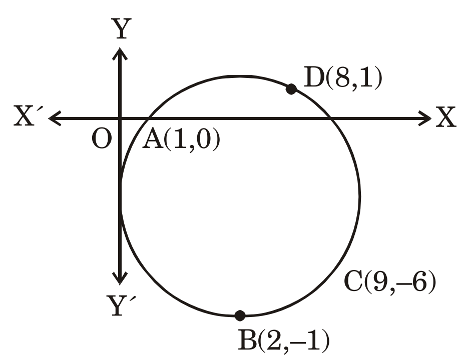
$$52 + 2g - 14f = 0$$

or $$26 + g - 7f = 0 \quad ...(5)$$

and subtracting (3) from (4), we get

$$12 + 2g + 16f = 0$$

$$\Rightarrow \quad 3 + 3g + 4f = 0 \quad ...(6)$$



Solving (5) and (6), we get

$$g = -5 \text{ and } f = 3$$

From (2), $1 - 10 + c = 0$

$\therefore \quad c = 9$

Therefore equation of circle passing through these point is

$$x^2 + y^2 - 10x + 6y + 9 = 0$$

Substituting the fourth point in the equation of this circle, we get

$(9)^2 + (-6)^2 - 10(9) + 6(-6) + 9 = 0$

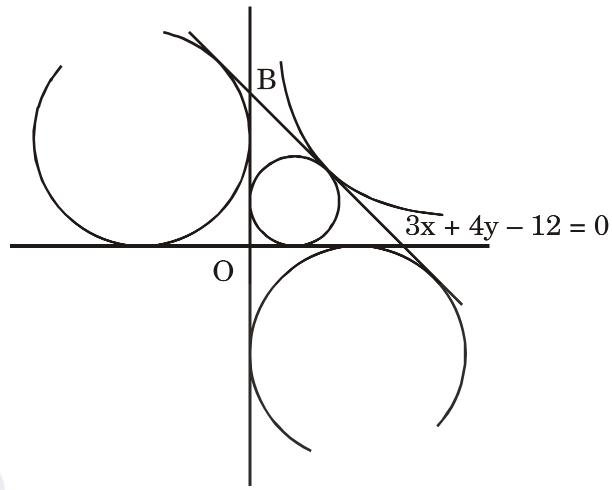
Hence point C (9, – 6) lies on the circle, that is, the four points are concyclic.

Illustration 9

Find the equation of all the circles touching x-axis, y-axis and line $3x + 4y - 12 = 0$.

Solution :

We can easily observe that there will be four circles touching x-axis, y-axis and the line $3x + 4y - 12 = 0$. The centre of any such circle must be of the form $(\pm r, \pm r)$ where r is the radius of the circle. If we want to determine the circles whose centres lie in first quadrant and which touch x-axis, y-axis and the line $3x + 4y - 12 = 0$ then centres must be (r, r) $(r > 0)$ and perpendicular distance of (r, r) from the line $3x + 4y - 12 = 0$ must also be r.



$\Rightarrow \quad \left|\frac{3r + 4r - 12}{5}\right| = r$

$\Rightarrow \quad 7r - 12 = 5r,\ 7r - 12 = -5r$

$\Rightarrow \quad r = 6,\ r = 1$

Thus two such circles are

$(x - 6)^2 + (y - 6)^2 = 6^2,\ (x - 1)^2 + (y - 1)^2 = 1^2$

Next in the second quadrant the centre of such a circle must be $(-r, r)$ and $\left|\frac{-3r + 4r - 12}{5}\right| = r$

$\Rightarrow \quad r = -3,\ r = 2$

The value $r = -3$ is inadmissible since $r > 0$. Thus the equation of circle touching axes and the line $3x + 4y - 12 = 0$, in the second quadrant is $(x + 2)^2 + (y - 2)^2 = 2^2$

In the third quarant $\left|\frac{-3r - 4r - 12}{5}\right| = r$

$\Rightarrow \quad r =,\ r = -6$

$\Rightarrow$ Both values are inadmissible.

Thus there is no such circle in the third quadrant.

Finally in the fourth quadrant

$\left|\frac{3r - 4r - 12}{5}\right| = r \quad \Rightarrow \quad r = 3$

The circle in the fourth quadrant must be $(x - 3)^2 + (y + 3)^2 = 3^2$

Second Method :

Note : We can solve the problem by trigonometry also. Denote the origin by C we have

$a = CB = 3,\ b = CA = 4$ and $c = AB = 5 \Rightarrow s = 6$

$$\Rightarrow \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = 6$$

Now $r = \dfrac{\Delta}{s} = \dfrac{6}{6} = 1$

$$r_1 = \frac{\Delta}{s-a} = \frac{6}{6-3} = 2$$

$$r_2 = \frac{\Delta}{s-b} = \frac{6}{6-4} = 3$$

$$r_2 = \frac{\Delta}{s-c} = \frac{6}{6-5} = 6$$

Since radii of such circles are determined the equations can be immediately written down since circle touches both axes. You must note that the four circles obtained are the inscribed circle (the one with the smallest radius) and the three described circles of the triangle formed by co-ordinate axes and the line $3x - 4y - 12 = 0$

Illustration 10

Find the area of an equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution :

Given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

Let O be the centre and ABC be an equilateral triangle inscribed in the circle (1).

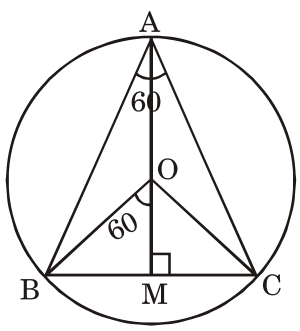
$$O \equiv (-g, -f)$$

and $OA = OB = OC = \sqrt{g^2 + f^2 - c}$ ...(2)

In ΔOBM, $\sin 60° = \dfrac{BM}{OB}$

$\Rightarrow \quad BM = OB \sin 60 = (OB)\dfrac{\sqrt{3}}{2}$

$BC = 2BM = \sqrt{3}\,(OB)$ ...(3)



Area of $\Delta ABC = \dfrac{\sqrt{3}}{4}(BC)^2$

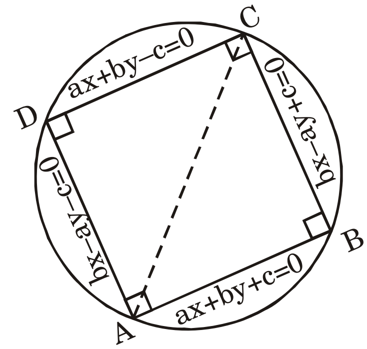
$= \dfrac{\sqrt{3}}{4} 3(OB)^2$ (from (3))

$= \dfrac{3\sqrt{3}}{4}\left(g^2 + f^2 - c\right)^2$ sq. units.

Illustration 11

Find the equation of the circumcircle of the quadrilateral formed by the four lines $ax + by \pm c = 0$ and $bx - ay + c = 0$

Solution :



The given lines can be re-written as

$$ax + by + c = 0 \quad ...(1)$$
$$ax + by - c = 0 \quad ...(2)$$
$$bx - ay + c = 0 \quad ...(3)$$
$$bx - ay - c = 0 \quad ...(4)$$

Equation (1) and (2) are parallel and equations (3) and (4) are also parallel.

Slope of (1) or (2) $= -\frac{a}{b} = m_1$ (say)

and Slope of (3) or (4) $= \frac{b}{a} = m_2$ (say)

Since $m_1 m_2 = -1$

Hence ABCD be a square and AC and BD are the diameters of the circle. After solving, we get

$$A \equiv \left(\frac{bc - ca}{a^2 + b^2}, \frac{bc + ca}{a^2 + b^2}\right) \text{ and } C \equiv \left(\frac{ac + bc}{a^2 + b^2}, \frac{ac - bc}{a^2 + b^2}\right)$$

$\therefore$ Equation of circle is $\left(x - \frac{bc - ca}{a^2 + b^2}\right)\left(x + \frac{ac + bc}{a^2 + b^2}\right) + \left(y + \frac{bc + ca}{a^2 + b^2}\right)\left(y - \frac{cc - bc}{a^2 + b^2}\right) = 0$

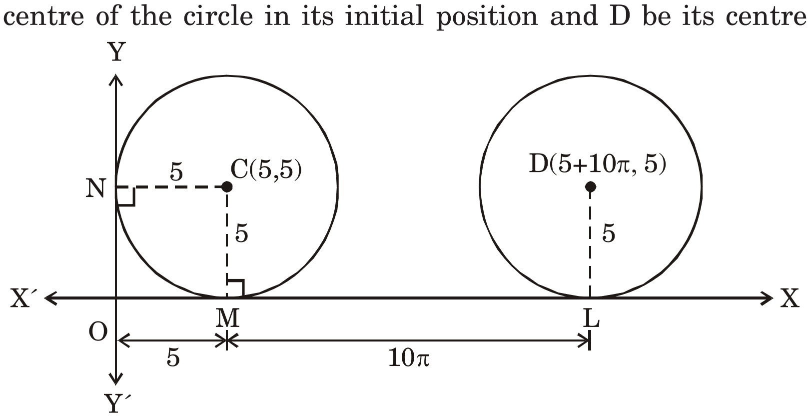
$$\Rightarrow \quad x^2 + y^2 + \left(\frac{2ac}{a^2 + b^2}\right)x + \left(\frac{2bc}{a^2 + b^2}\right)y = 0$$

Illustration 12

A circle of radius 5 units touches the co-ordinate axes in first quadrant. If the circle makes one complete roll on x-axis along the positive direction of x-axis, find its equation in the new position.

Solution :

Let C be the centre of the circle in its initial position and D be its centre in the new position.



Since the circle touches the co-ordinates axes in first quadrant and the radius of circle be 5 units.

$\therefore$ Centre of circle is (5, 5)

Moving length of circle = circumference of the circle

$$= 2\pi r = 2\pi (5) = 10\pi$$

Now centre of circle in new position is $(5 + 10\pi, 5)$ and radius is 5 units, therefore, its equation will be

$$(x - 5 - 10\pi)^2 + (y - 5)^2 = 5^2$$

or $x^2 + y^2 - 10(1 + 2\pi)x - 10y + 100\pi^2 + 100\pi + 25 = 0$

5. Position of a point with respect to circle

Let the equation of the circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $A(x_1, y_1)$ be any point.

Distance between the centre $(-g, -f)$ and A is given by $d = \sqrt{(x_1 + g)^2 + (y_1 + f)^2}$

$$d^2 - r^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + g^2 + f^2 - g^2 - f^2 + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = S_1$$

Clearly A lies inside, on or outside the circle as d <, =, or > i.e., $d^2 - r^2$ <, =, or < 0

Hence $S_1(A) < 0 \Rightarrow$ point lies inside the circle.

$S_1(A) = 0 \Rightarrow$ point lies on the circle.

$S_1(A) > 0 \Rightarrow$ point lies outside the circle.

Maximum and Minimum Distance of a Point from the Circle

Let any point $P(x_1, y_1)$ and circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(1)$$

The centre and radius of the circle are

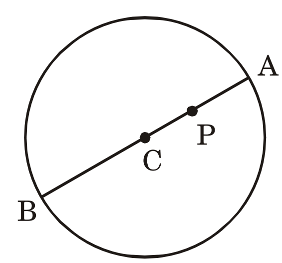
$$C(-g, -f) \text{ and } \sqrt{(g^2 + f^2 - c)} \text{ respectively}$$

Case I :

If P inside the circle

In this case $S_1 < 0$

$\therefore$ $r = \sqrt{(g^2 + f^2 - c)} = CA = CB$



The **minimum distance** of P from circle $= PA = CA - CP$

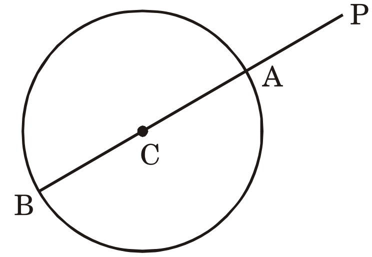
$= |r - CP|$

and the **maximum distance** of P from circle $= PB$

$= CB + CP = |r + CP|$

Case II :

If P outside the circle



In this case $S_1 > 0$

the **minimum distance** of P from circle

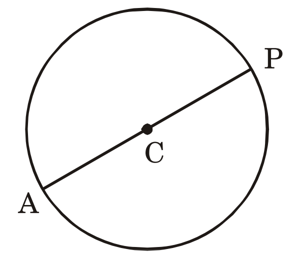
$$= PA = CP - CA = |CP - r|$$

and the **maximum distance** of P from the circle

$$= PB = CP + CB = |r + CP|$$

Case III :

If P on the circle



In this case $S_1 = 0$

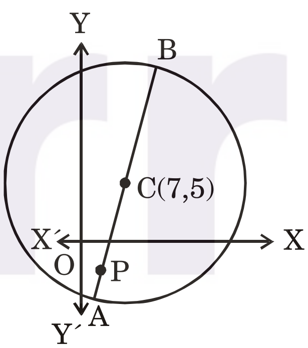
the **minimum distance** of P from the circle = 0

and the **maximum distance** of P from the circle = PA = 2r

Illustration 13

Find the shortest and largest distance from the point (2, – 7) to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$

Solution :



Let $S \equiv x^2 + y^2 - 14x - 10y - 151 = 0$

$\therefore$ $S_1 = (2)^2 + (-7)^2 - 14(2) - 10(-7) - 151$

$= -56 < 0$

$\therefore$ P (2, – 7) inside the circle

radius of the circle, $r = \sqrt{(7-2)^2 + (-5)^2 + 151} = 15$

$\therefore$ Centre of circle $\equiv (7, 5)$

$\therefore$ $CP = \sqrt{(7-2)^2 + (5+7)^2} = 13$

$\therefore$ **Shortest distance** $= PA = r - CP = 15 - 13$

$= 2$

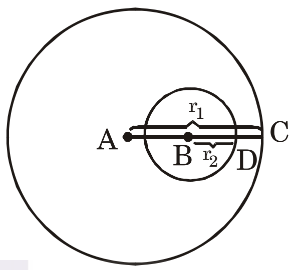
and **Largest distance** $= PB = r + CP = 15 + 13 = 28$

6. Contact of Two Circles

Working Rule :

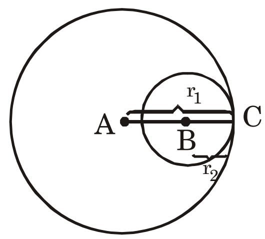
The two circles having centres at A (x_1, y_1) and B (x_2, y_2) and radii r_1 and r_2 respectively touch each other.

$AB < r_1 - r_2$		$r_1 - r_2 < AB < r_1 + r_2$		$AB > r_1 + r_2$
Smaller circle lies inside the bigge circle	AB $r = r_1 - r_2$ ↓ circles touch internally	circle intersects in two points	AB $= r_1 + r_2$ ↓ circles touch externally	One circle lies out side the other

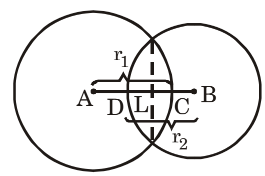


$r_1 - r_2 = AB + DC$

$\therefore$ $AB < r_1 - r_2$



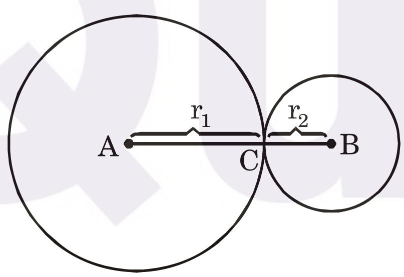
$AB = r_1 - r_2$



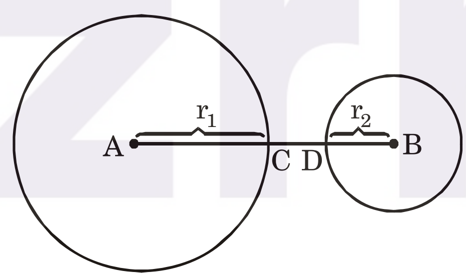
$r_1 + r_2 = AC + CB + DC$

$= AB + DC$

$\therefore$ $AB < r_1 + r_2$



$AB = r_1 + r_2$



$AB = r_1 + r_2 + CD$

$AB > r_1 + r_2$

Illustration 14

Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.

Solution :

Given circles are $x^2 + y^2 - 2x - 4y = 0$ ...(1)

and $x^2 + y^2 - 8y - 4 = 0$ ...(2)

Let A and B be the centres and r_1 and r_2 the radii of circles (1) and (2) respectively, then

$A \equiv (1, 2), B \equiv (0, 4), r_1 = \sqrt{5}, r_2 = 2\sqrt{5}$

Now $AB = \sqrt{(1-0)^2 + (2-4)^2} = \sqrt{5}$

and $r_1 + r_2 = 3\sqrt{5}. |r_1 - r_2| = |\sqrt{5} - 2\sqrt{5}| = \sqrt{5}$

Thus $AB = |r_1 - r_2|$, hence the two circles touch each other internally.

Illustration 15

Prove that the circle $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

Solution :

Given circles are

$$x^2 + y^2 + 2ax + c^2 = 0 \quad ...(1)$$

and

$$x^2 + y^2 + 2by + c^2 = 0 \quad ...(2)$$

Let A and B be the centres of circles (1) and (2) respectively and r_1 and r_2 be their radii, then

$A \equiv (-a, 0), B \equiv (0, b), r_1 = \sqrt{a^2 - c^2}, r_2 = \sqrt{b^2 - c^2}$

The two circles (1) and (2) will touch each other externally or internally according as $AB = r_1 + r_2$ or $AB = |r_1 - r_2|$

i.e. $AB^2 = (r_1 + r_2)$ or $AB^2 = (r_1 - r_2)^2$

Thus the two circles will touch each other if

$$AB^2 = (r_1 \pm r_2)^2 \text{ or } a^2 + b^2 = r_1^2 + r_2^2 \pm 2r_1r_2$$

or $a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}$

or $2c^2 = \pm 2\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}$

$C^2 = \pm\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}$

or $c^4 = (a^2 - c^2)(b^2 - c^2)$ or $c^4 = a^2b^2 - c^2b^2 - a^2c^2 + c^4$

or $c^2b^2 + a^2c^2 = a^2b^2$

or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ [dividing by $a^2b^2c^2$]

Illustration 16

Prove that $x^2 + y^2 = a^2$ and $(x - 2a)^2 + y^2 = a^2$ are two equal circles touching each other. Find the equation of circle (or circles) of equal radius touching both the circles.

Solution :

Given circles are

$x^2 + y^2 = a^2$ ...(1)

and $(x - 2a)^2 + y^2 = a^2$ ...(2)

Let A and B be the centres and r_1 and r_2 the radii of the two circle (1) and (2) respectively, then $A \equiv (0, 0)$, $B \equiv (2a, 0)$, $r_1 = a$, $r_2 = a$

Now $AB = \sqrt{(0-2a)^2 + 0^2} = 2a = r_1 + r_2$

Hence the two circles touch each other externally.

Let the equation of equal circle touching circles (1) and (2) be $(x - \alpha)^2 + (y - \beta)^2 = a^2$ ...(3)

Its centre C is (α, β) and radius $r_3 = a$.

Since (3) touches (1).

$\therefore$ $AC = r_1 + r_3 = 2a$. [Here $AC \neq |r_1 - r_3|$ as $r_1 - r_3 = a - a = 0$]

or $AC^2 = 4a^2$ or $\alpha^2 + \beta^2 = 4a^2$ ...(4)

Again since circle (3) touches circle (2)

$\therefore$ $BC = r_2 + r_3$ or $BC^2 = (r_2 + r_3)^2$

$\therefore$ $(2a - \alpha)^2 + \beta^2 = (a + a)^2$ or $\alpha^2 + \beta^2 - 4a\alpha = 0$

or $4a^2 - 4a\alpha = 0$ [from (4)]

$\therefore$ $\alpha = a$ and from (4), $\beta = \pm\sqrt{3}\,a$

Thus required circles are $(x - a)^2 + \left(y \mp \sqrt{3}a\right)^2 = a^2$

or $x^2 + y^2 - 2ax \mp 2\sqrt{3}\,ay + 3a^2 = 0$

7. THE INTERSECTION OF A LINE AND A CIRCLE

Any line will cut a circle either

(a) at two distinct points or

(b) at two coincident points or

(c) at two imaginary points.

In the second case the line will be tangent to the circle. Let us observe the intersection of the line $y = mx + c$ and the circle $x^2 + y^2 = a^2$. The x-ordinates of points of intersection of the line and the circle are given by the equation

$$x^2 + (mx + c)^2 = a^2$$

$$\Rightarrow \quad (1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$$

In general, there are 3 cases arises. When circle and a line intersects

Case I :

There are two different real points of intersection.

i.e., $\Delta > 0$

$\Rightarrow \quad 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$

$\therefore \quad a^2(1 + m^2) > c^2$

or $\quad a > \left|\dfrac{c}{\sqrt{1 + m^2}}\right|$

Case II :

There are two coincident points of intersection.

i.e., $\Delta = 0$

$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$

$\Rightarrow \quad a^2(1 + m^2) = c^2$

$\Rightarrow \quad a = \left|\dfrac{c}{\sqrt{1 + m^2}}\right|$

$\Rightarrow \quad \boxed{c = \pm a\sqrt{1 + m^2}}$

This is the required condition of tangency.

$\therefore$ the equation the tangents are $y = mx + a\sqrt{1 + m^2}$

and $y = mx - a\sqrt{1 + m^2}$

Case III :

There is no real point of intersection

i.e., $\Delta < 0$

$\Rightarrow \quad 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) < 0$

$\Rightarrow \quad m^2c^2 - (1 + m^2)(c^2 - a^2) < 0$

$\Rightarrow \quad m^2c^2 - c^2 - m^2c^2 + a^2 + a^2m^2 < 0$

$\Rightarrow \quad a^2(1 + m^2) < c^2$

$\Rightarrow \quad a^2 < \dfrac{c^2}{(1 + m^2)}$

$\Rightarrow \quad a < \left|\dfrac{c}{\sqrt{1 + m^2}}\right|$

Thus the line $y = mx + c$; will

(i) cut the circle $x^2 + y^2 = a^2$ at two distinct points if $a^2(1 + m^2) > c^2$

(ii) touch the circle if, $a^2(1 + m^2) = c^2$

(iii) not cut the circle at all if $a^2(1 + m^2) < c^2$.

Note :

(1) The line $y = mx \pm a\sqrt{1 + m^2}$ will be a tangent to the circle $x^2 + y^2 = a^2$ for all values of m.

(2) The condition of tangency can also be obtained by equating perpendicular distance of the centre (0, 0) than the line $y - mx - c = 0$ to the radius of the circle.

(3) In case the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ we may shift the origin to the centre of the circle and apply the results described above.

Another Method :

Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and $L = lx + my + n = 0$

Let r be the radius of the circle and p be the length of the perpendicular drawn from the centre $(-g, -f)$ on the line L. Then

(i) line intersect the circle in two distinct points if $p < r$.

(ii) line touch the circle if $p = r$

(iii) line neither intersects nor touches the circle i.e., passes outside the circle if $p > r$.

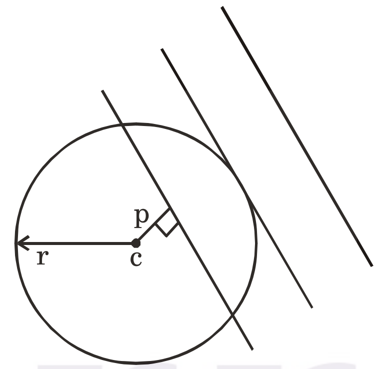


Illustration 17

Find the values of k for which the line $3x - 4y - k = 0$ will touch the circle $x^2 + y^2 - 2x - 4y + h = 0$. Find the point of contact also.

Solution :

We will solve the problem by two methods. The circle is

$$(x - 1)^2 + (y - 2)^2 = 1$$

Shifting the origin to (1, 2) the circle becomes $x^2 + y^2 = 1$ and the line becomes

$$3x - 4y - (5 + k) = 0$$

Now applying $c^2 = a^2(1 + m^2)$

$$\left(-\frac{5+k}{4}\right)^2 = 1\left(1 + \frac{9}{16}\right) \Rightarrow k = 0,\ k = 10$$

If $k = 0$ then $c = -\frac{5}{4}$

and the point of contact is $\left(-\frac{3/4}{-5/4}, \frac{1}{-5/4}\right)$ or, is $\left(\frac{3}{5}, \frac{-4}{5}\right)$

Now with reference to old origin this point will be $\left(\frac{3}{5}+1, \frac{-4}{5}+2\right)$ or will be $\left(\frac{8}{5}, \frac{6}{5}\right)$

Alternatively we can proceed as follows. For intersection of line and circle

$$x^2+\left(\frac{3x-k}{4}\right)^2-2x-4\left(\frac{3x-k}{4}\right)+4=0$$

$\Rightarrow \quad 25x^2 - (6k + 80)x + (k + 8)^2 = 0$

$$\Rightarrow \quad x = \frac{6k+80 \pm \sqrt{(6k+80)^2-100(k+8)^2}}{50}$$

$$= \frac{3k+40 \pm \sqrt{(8k+80)(-2k)}}{25}$$

In case the line happens to be tangent we must have only one value of x.

$\Rightarrow$ Discriminant of the quadratic = 0

$\Rightarrow$ $k = 0, k = -10$

Illustration 18

Show that the line 3x – 4y – c = 0 will meet the circle having centre at (2, 4) and the radius 5 in real and distinct points if – 35 < c < 15.

Solution :

Given line is $3x - 4y - c = 0$ ...(i)

Centre of given circle circle is (2, 4) and its radius is 5, therefore its equation will be

$(x - 2)^2 + (y - 4)^2 = 5^2$

or $x^2 + y^2 - 4x - 8y - 5 = 0$ ...(ii)

From (i), $y=\frac{1}{4}(3x-c)$. Putting the value of y in (ii), we get

$$x^2+\frac{1}{16}(3x-c)^2-4x-8\frac{1}{4}(3x-c)-5=0$$

or $16x^2 + 9x^2 - 6cx + c^2 - 64x - 96x + 32c - 80 = 0$

or $25x^2 - 2(80 + 3c)\,x + c^2 + 32c - 80 = 0$ ...(iii)

Line (i) will meet the circle (ii) in real and distinct points if discriminant of equation (iii) > 0

i.e., if $4(80 + 3c)^2 - 100\,(c^2 + 32c - 80) > 0$ or $(80 + 3c)^2 - 25(c^2 + 32c - 80) > 0$

or $6400 + 9c^2 + 480c - 25c^2 - 800c + 2000 > 0$

or $-16c^2 - 320c + 8400 > 0$ or $16c^2 + 320c - 8400 < 0$

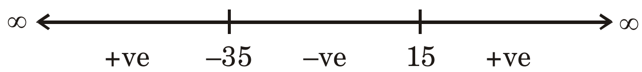
or $c^2 + 20c - 525 < 0$

sign scheme for $c^2 + 20c - 525$:

When $c^2 + 20c - 525 = 0$

$$c = \frac{-20 \pm \sqrt{400 + 2100}}{2} = \frac{-20 \pm 50}{2} = 35, 15$$

Therefore, sign scheme for $c^2 + 20c - 525$ is as follows



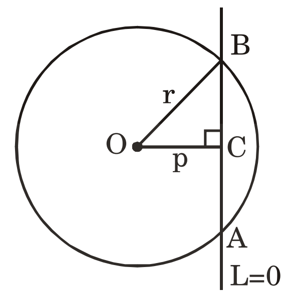
$\therefore \quad c^2 + 20c - 525 < 0 \; -35 < c < 15$

Length of the chord

Let O be the centre of the circle S = 0 and the line of the chord AB be L= 0

Let p = length of ⊥ from centre to the line L = 0

Then AB = length of the chord = $2\sqrt{(r^2 - p^2)}$. (by Pythagoras theorem)



TANGENT

In Geometry the tangent at any point of a circle is defined to be a straight line which meets the circle there, but, being produced, does not cut it; this tangent is shown to be always perpendicular to the radius drawn from the centre to the point of contact.

8. DIFFERENT FORMS OF THE EQUATION OF TANGENT

8.1.1. Point form

The equation of the tangent at a point P (x, y) to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\boxed{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$$

for equation of circle in standard form the equation becomes

$$\boxed{xx_1 + yy_1 = a^2}$$

Note : For equation of tangent of circle at (x_1, y_1), substitute xx_1 for x^2, yy^1 for x^2, yy_1 for y^2, $\frac{x + x_1}{2}$ for x, $\frac{y + y_1}{2}$ for y and $\frac{xy_1 + x_1 y}{2}$ for xy and keep the constant as such

8.2. Parametric form :

The equation of tangent to the circle $x^2 + y^2 = a^2$ at the point $(a \cos \theta, a \sin \theta)$ is

$$\boxed{x \cos \theta + y \sin \theta = a}$$

Note : The equation of the tangent to the circle $(x - a)^2 + (y - b)^2 = r^2$ at the point $(a + r \cos \theta, b + r \sin \theta)$ is

$$(x - a) \cos \theta + (y - b) \sin \theta = r.$$

8.3. Slope form :

The equation of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{(1 + m^2)}$ and the co-ordinates of the point of contact are $\left(\pm \frac{am}{\sqrt{(1+m^2)}}, \mp \frac{a}{\sqrt{(1+m^2)}}\right)$

We already have done the proof for this.

Illustration 19

Find the condition that the straight line $lx + my + n = 0$ may touch circle $x^2 + y^2 = a^2$ and also find the co-ordinates of the point of contact.

Solution :

given line is $lx + my + n = 0$ ...(1)

and given circle is $x^2 + y^2 = a^2$ ...(2)

line (1) will touch the circle (2) if the length of the perpendicular from its centre (0, 0) to line (1) = radius of circle (2)

i.e. if $\frac{|l.0 + m.0 + n|}{\sqrt{l^2 + m^2}} = a$

or $n^2 = a^2(l^2 + m^2)$ This is the required condition.

Second part :

Let line (1) be tangent to circle (2) at (α, β).

Now equation of the tangent to circle (2) at (α, β) is $x\alpha + y\beta - a^2 = 0$ ...(3)

Since (1) and (3) are equations of the same straight line, therefore comparing the coefficients, we get

$$\underset{(i)}{\frac{\alpha}{l}} = \underset{(ii)}{\frac{\beta}{m}} = \underset{(iii)}{\frac{-a^2}{n}}$$

From (i) and (iii), $\alpha = -\frac{la^2}{n}$ and from (ii) and (iii), $\beta = -\frac{ma^2}{n}$

Thus point of contact of line (1) and circle (2) is $\left(-\frac{la^2}{n}, -\frac{ma^2}{n}\right)$

Note : Condition for tangency can be obtained by putting $x = -\frac{la^2}{n}$, $y = -\frac{ma^2}{n}$ in the equation of the circle because point of contact lies on the circle.

Illustration 20

Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point (1, – 2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ and find its point of contact.

Solution :

Given circles are $x^2 + y^2 = 5$ ...(1)

and $x^2 + y^2 - 8x + 6y + 20 = 0$ ...(2)

Given point is (1, – 2).

Now equation of the tangent to circle (1), at (1, – 2) will be

$x \cdot 1 + y(-2) - 5 = 0$ or $x - 2y - 5 = 0$ ...(3)

Centre of circle (2) is C (4, – 3) and its radius is $\sqrt{5}$

Now length of the perpendicular from C(4, – 3) to line (3)

$$= \frac{|4 - 2(-3) - 5|}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5} = \text{radius of circle (2)}$$

Hence line (3) also touches circle (2).

Second part :

Let (α, β) be the point of contact of line (3) and circle (2).

Now equation of the tangent to circle (2) at (α, β) is

$$x\alpha + y\beta - 8\left(\frac{x+\alpha}{2}\right) + 6\left(\frac{y+\beta}{2}\right) + 20 = 0$$

or $(\alpha - 4)x + (\beta + 3)y - (4\alpha - 3\beta - 20) = 0$ ...(4)

Comparing equations (3) and (4), we get

$$\underset{\text{(i)}}{\frac{\alpha - 4}{1}} = \underset{\text{(ii)}}{\frac{\beta + 3}{-2}} = \underset{\text{(iii)}}{\frac{4\alpha - 3\beta - 20}{5}}$$

From (i) and (ii), we get $2\alpha + \beta = 5$ ...(5)

and from (i) and (iii), we get $\alpha + 3\beta = 0$ ...(6)

Solving (5) and (6), we get $\alpha = 3$, $\beta = -1$

Hence point of contact of circle (2) and line (3) is (3, – 1).

Illustration 21

Prove that the equation $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$, where λ is a parameter, represents a family of circles passing through two fixed points A and B on the x-axis. Also find the equation of that circle of the family, the tangents to which at A and B meet on the line $x + 2y + 5 = 0$

Solution :

Given equation of the family of circles is

$$x^2 + y^2 - 2x - 2\lambda y - 8 = 0 \quad \text{...(1)}$$

It can be written as $x^2 + y^2 - 2x - 8 - 2\lambda y = 0$ ...(2)

Clearly (2) is the equation of circles passing through the point of intersection of circle

$$x^2 + y^2 - 2x - 8 = 0$$

and the line $y = 0$ ...(3)

Putting $y = 0$ in (2), we get $x^2 - 2x - 8 = 0$ $\therefore$ $x = 4, -2$

Let A ≡ (4, 0) and B ≡ (– 2, 0). Thus (1) represents a family of circles passing through two fixed points A (4, 0) and B (– 2, 0) on the x-axis.

Now equation of tangents to circle (1) at (4, 0) and B (– 2, 0) are

$$4x + y.0 - (x + 4) - \lambda(y + 0) - 8 = 0 \text{ or } 3x - \lambda y - 12 = 0 \quad \text{...(4)}$$

and $-2x + y.0 - (x - 2) - \lambda(y + 0) - 8 = \text{ or } -3x - \lambda y - 6 = 0$ ...(5)

Solving (4) and (5), we get $x = 1$, $y = -\dfrac{9}{\lambda}$

If $\left(1, -\dfrac{9}{\lambda}\right)$ lies on line $x + 2y + 5 = 0$

then $1 - \dfrac{18}{\lambda} + 5 = 0$ or $\lambda = 3$

Putting the value of λ in (1), we have the equation of the required circle as

$$x^2 + y^2 - 2x - 6y - 8 = 0$$

Illustration 22

Find the equations of the tangents to the circle $x^2 + y^2 = 9$, which

(i) are parallel to the line $3x + 4y - 5 = 0$

(ii) are perpendicular to the line $2x + 3y + 7 = 0$

(iii) make an angle of 60 with the x-axis.

Solution :

(i) Slope of $3x + 4y - 5 = 0$ is $-\dfrac{3}{4}$

Let $m = -\dfrac{3}{4}$

and equation of circle is $x^2 + y^2 = 9$

$\therefore$ Equations of tangents $y = -\frac{3}{4}x \pm 3\sqrt{\left(1 + \left(-\frac{3}{4}\right)^2\right)}$

$\Rightarrow$ $4y = -3x \pm 15$

or $3x + 4y \pm 15 = 0$

(ii) slope of $2x + 3y + 7 = 0$ is $-\frac{2}{3}$

$\therefore$ Slope of perpendicular to $2x + 3y + 7 = 0$ is $\frac{3}{2} = m$ (say)

and given circle is $x^2 + y^2 = 9$

$\therefore$ Equations of tangents perpendicular to $2x + 3y + 7 = 0$ is

$$y = \frac{3}{2}x \pm 3\sqrt{1 + \left(\frac{3}{2}\right)^2}$$

$\Rightarrow$ $2y = 3x \pm 3\sqrt{13}$

or $3x - 2y \pm 3\sqrt{13} = 0$

(iii) Since tangent make an angle 60 with the x-axis

$\therefore$ $m = \tan 60° = \sqrt{3}$

and given circle $x^2 + y^2 = 9$

$\therefore$ Equation of tangents $y = \sqrt{3}x \pm 3\sqrt{1 + \left(\sqrt{3}\right)^2}$

or $\sqrt{3}\,x - y \pm 6 = 0$

Alternative Method

(i) Let tangent parallel to $3x + 4y - 5 = 0$ is

$3x + 4y + \lambda = 0$ ...(1)

and circle $x^2 + y^2 = 9$

then perpendicular distance from (0, 0) on (2) = radius

$$\frac{|\lambda|}{\sqrt{(3^2 + 4^2)}} = 3$$

or $|\lambda| = 15$

$\therefore$ $\lambda = \pm 15$

From (1), equations of tangents are

$3x + 4y \pm 15 = 0$

(ii) Let tangent perpendicular to $2x + 3y + 7 = 0$ is

$$3x - 2y + \lambda = 0 \qquad ...(2)$$

and circle $x^2 + y^2 = 9$

then perpendicular distance from (0, 0) on (2) = radius

$$\frac{|\lambda|}{\sqrt{3^2 + (-2)^2}} = 3$$

or $|\lambda| = 3\sqrt{13}$

or $|\lambda| = \pm 3\sqrt{13}$

From (2), equations of tangents are $3x - 2y \pm 3\sqrt{13} = 0$

(iii) Let equation of tangent which makes an angle of 60 with the x-axis is

$$y = \sqrt{3}x + c \qquad ...(3)$$

or $\sqrt{3}x - y + c = 0$

and circle $x^2 + y^2 = 9$

then perpendicular distance from (0, 0) to (3) = radius

$$\frac{|c|}{\sqrt{(3)^2 + (-1)^2}} = 3$$

or $|c| = 6$

or $c = \pm 6$

From (3), equations of tangent are $\sqrt{3}x - y \pm 6 = 0$

5. Length of Tangent

I. The length of the tangent drawn from a given external point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is given by $\sqrt{x_1^2 + y_1^2 - a^2}$

Proof :

Equation of given circle is $x^2 + y^2 = a^2$ ...(1)

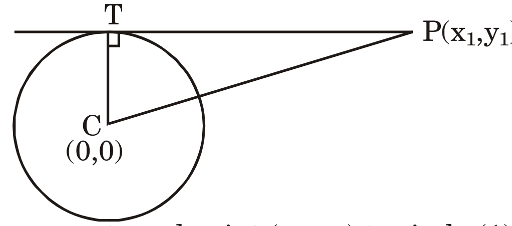
Its centre is C (0, 0) and radius is a. Let P (x_1, y_1) be the given point. From P let PT be a tangent to the circle touching the circle at T. Then PT will be the length of the tangent. Join C, P and C, T since PT $\perp$ CT hence from right angled ΔCTP, $CT^2 + PT^2 = CP^2$

or $PT^2 = CP^2 - CT^2$ ...(1)

But CT = the radius of the circle = a

and $CP^2 = (x_1 - 0)^2 + (y_1 - 0)^2 = x_1^2 + y_1^2$

Therefore, from (1) $PT^2 = x_1^2 + y_1^2 - a^2$



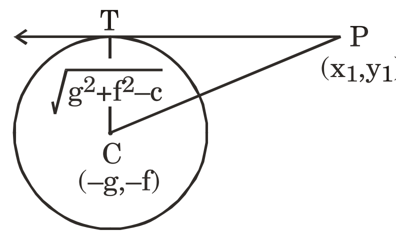
or $\boxed{PT = \sqrt{x_1^2 + y_1^2 - a^2}}$

This is the required length of the tangent drawn from an external point (x_1, y_1) to circle (1).

II. The length of the tangent drawn from a given external point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ given by $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ ...(1)

The equation of given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Its centre is C $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.



Let P (x_1, y_1) be the given point. From P Let PT be a tangent to the circle touching the circle at T. Then PT will be the length of the tangent. Join C, T and C, P. Since CT ⊥ PT.

∴ from right angled ΔCTP,

$$CT^2 + PT^2 = CP^2, \text{ or } PT^2 = CP^2 - CT^2 \quad ...(2)$$

But CT = the radius of the circle = $\sqrt{g^2 + f^2 - c}$

and $CP^2 = (x_1 - (-g))^2 + \{y_1 - (-f)\}^2 = (x_1 + g)^2 + (y_1 + f)^2$

$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + g^2 + f^2$

Therefore, from (2), we have

$$PT^2 = (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + g^2 + f^2) - (\sqrt{g^2 + f^2 - c})^2$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + g^2 + f^2 - g^2 - f^2 + c$$

$$\Rightarrow \quad PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

This is the required length of the tangent drawn from an external point (x_1, y_1) to the given circle (1).

Tip : If $S \equiv 0$ is the equation of given circle and $s_1 \equiv 0$ is the equation achieved by butting (x, y_1) in s, then the length o tangent from $(x_1\ y_1)$ is given by $\sqrt{S_1}$.

Illustration 23

If the length of tangent from (f, g) to the circle $x^2 + y^2 = 6$ be twice the length of the tangent from (f, g) to circle $x^2 + y^2 + 3x + 3y = 0$ then will $f^2 + g^2 + 4f + 4g + 2 = 0$?

Solution :

According to the question

$$\sqrt{(g^2 + f^2 - 6)} = 2\sqrt{(f^2 + g^2 + 3f + 3g)}$$

On squaring $\quad g^2 + f^2 - 6 = 4f^2 + 4g^2 + 12f + 12g$

or $\quad 3f^2 + 3g^2 + 12f + 12g + 6 = 0$

or $\quad f^2 + g^2 + 4f + 4g + 2 = 0$

which is true $\therefore$ yes.

Illustration 24

Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact find the length of OA.

Solution :

The given equation represents lines y = x and y = 2x

if 2 φ is the angle between them

$$\tan 2\varphi = \frac{2-1}{1+2} = \frac{1}{3} = \frac{2\tan\varphi}{1-\tan^2\varphi} \Rightarrow \tan\varphi = -3 \pm \sqrt{10}$$

But, as centre lies in the first quadrant

and $0 < \varphi < \frac{\pi}{4}$

$\tan\varphi = -3 + \sqrt{10}$ only. Now OA = 3 cot φ = $\frac{3}{-3+\sqrt{10}}$

Illustration 25

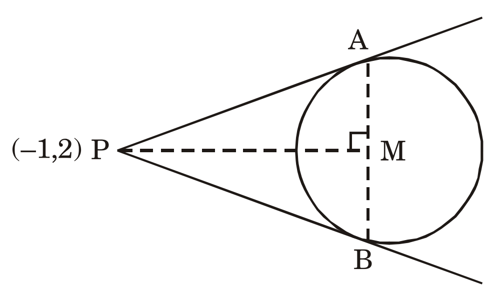
Tangents PA and PB are drawn from P (– 1, 2) to the circle $x^2 + y^2 - 2x - 4y + 2 = 0$

(i) Find the lengths PA (or PB)

(ii) Find the area of ΔPAB

(iii) Find the equation of PA and PB

Solution :



(i) PA^2 or PB^2

= value of circles equation for x = – 1 and y = 2

= 1 + 4 + 2 – 8 = 1 ⇒ PA = PB = 1

(ii) Equation of chord of contact AB must be

x (– 1) + y (2) – 1 (x – 1) – 2 (y + 2) + 2 = 0 or 2x + 1 = 0

Now PM = ⊥ distance of P (– 1, 2) from AB = $\frac{2(-1)+1}{\sqrt{2^2+0^2}} = \frac{1}{2}$

$\Rightarrow$ $AM = \sqrt{PA^2 - PM^2} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

$\Rightarrow$ Area of ΔPAB = $\frac{1}{2}$ base. height = $\frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4}$ square units.

(iii) Equation of any line through P (– 1, 2) is y – 2 = m (x + 1) or y – mx – 2 – m = 0

If this happens to be tangent to the given circle the length from the centre (1, 2) on the line must be equal to the radius $\sqrt{3}$.

$\Rightarrow$ $\frac{|2-m-2-m|}{\sqrt{1+m^2}} = \sqrt{3} \Rightarrow 4m^2 = 3m^2 + 3 \Rightarrow m = \pm\sqrt{3}$

Thus the equations of tangents PA and PB are $y - 2 = \sqrt{3}(x+1)$ and $y - 2 = -\sqrt{3}(x+1)$

Note : Part (iii) can also be done by using $S S_1 = T^2$

We already know the values of S_1 and T in this problem

$S_1 = 1$ (since (length)^2 of PA or PB)

T = 2x + 1 (equation of chord of contact)

Therefore combined equation of PA nd PB is

$(x^2 + y^2 - 2x - 4y + 2) . 1 = (2x + 1)^2$ or $-3x^2 + y^2 - 6x - 4y + 1 = 0$

Let $y^2 - 3x^2 - 6x - 4y + 1 \left(y - \sqrt{3}x - c_1\right)\left(y + \sqrt{3}x + c_2\right)$

On comparing coeffs. of x, y and constant terms, we get

$-6 = -c_2\sqrt{3} - c_1\sqrt{3}$, $-4 = c_2 - c_1$, $1 = c_1 c_2$

On solving first two we get $c_1 = 2 - \sqrt{3}, c_2 = 2 + \sqrt{3}$

These values satisfy the third equation $c_1 c_2 = 1$ ⇒ The assumption (1) is justified

$\Rightarrow$ Lines are $y = \sqrt{3}x + 2 + \sqrt{3}$ and $y = -\sqrt{3}x + 2 - \sqrt{3}$

Illustration 26

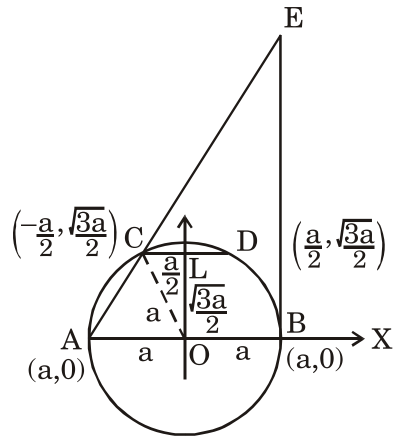
AB is a diameter of a circle. CD is a chord parallel to AB and 2 CD = AB. The tangent at B meets the line AC produced at E. Prove that AE = 2AB.

Solution :

Let O be the centre of the circle which is taken as the origin. Let a be the radius of the circle.

Now $A \equiv (-a, 0)$, $B \equiv (a, 0)$

Since $CD \parallel AB$ and $2CD = AB$



Also $CL = \frac{CD}{2} = \frac{AB}{4} = \frac{2a}{4} = a/2$

In ΔOLC

$$OL = \sqrt{OC^2 - CL^2} = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}\,a}{2}$$

$$\therefore \quad C \equiv \left(-\frac{a}{2}, \frac{\sqrt{3}}{2}a\right), D \equiv \left(\frac{a}{2}, \frac{\sqrt{3}}{2}a\right)$$

Now equation of circle is $x^2 + y^2 = a^2$ ...(1)

and equation of tangent at B(a, 0) is

$xa + y.0 = a^2$ or $x = a$ ...(2)

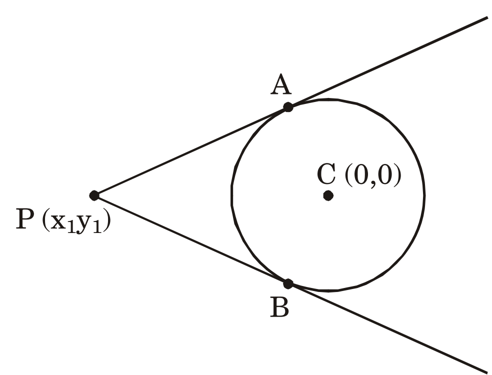
Equ. of AC is $y - 0 = \dfrac{0 - \frac{\sqrt{3}}{2}a}{-a + \frac{a}{2}}(x + a)$ or $y = \sqrt{x}(x + a)$ ...(3)

Solving (2) and (3), we get $x = a$, $y = 2\sqrt{3}a$ $\therefore$ $E \equiv (a, 2\sqrt{3}a)$

Now $AE \equiv \sqrt{(-a - a)^2 + (0 - 2\sqrt{3}a)^2} = 4a = 2.2a = 2.AB$

10. Equation of pair of tangents

The equation of pair of pair of tangents drawn from an external point P $(x_1\ y_1)$ to the circle $x^2 + y^2 = a^2$ is given by



$$\boxed{(x^2 + y^2 - a^2)\ (x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)}$$

OR we can say $\boxed{\mathbf{SS_1 = T^2}}$

where S $\equiv$ equation of circle

S_1 $\equiv$ value after putting $(x_1\ y_1)$ in equation of circle

T $\equiv$ equation of tangent at point $(x_1\ y_1)$

Note : If the equation of circle is given in general form then also the condition $SS_1 = T^2$ is valud for the equation of circle, $x^2 + y^2 + 2gx + 2fy + c = 0$, the equation becomes

$$(x^2 + y^2 + 2gx + 2fy + c)\ (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)$$

$$= (xx_1 + yy_1 + g\,(x + x_1) + f\,(y + y_1))^2 + c$$

Note that the equation of tangent is also with respect to the general equation of circle.

11. Director circle

The locus of the point of intersection of two perpendicular tangents to a given conic is known as its director circle.

The equation of the director circle of the circle $x^2 + y^2 = a^2$ is $\mathbf{x^2 + y^2 = 2a^2}$

12. Normal To the Circle

The normal at any point P of a curve is the straight line which passes through P and is perpendicular to the tangent at P.

Generally there is no need to remember the formula for normal to circle. We follow the following steps to find the equation of normal.

Step 1 : Write the equation of tangent to the given circle at $(x_1\ y_1)$

Step 2 : Write the equation of a line which is perpendicular to the above line & passes through $(x_1\ y_1)$. The equation obtained is the equation of normal.

To find the equation to the normal at the point (x', y') of (1), the circle,

$$x^2 + y^2 = a^2$$

and (2), the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

(1) The tangent at (x', y') is

$$xx' + yy' = a^2$$

i.e.

$$y = \frac{x'}{y'}x + \frac{a^2}{y'}$$

The equation to the straight line passing through (x', y') perpendicular to this tangent is,

$$y - y' = m\,(x - x')$$

Where

$$m \times \left(-\frac{x'}{y'}\right) = -1$$

i.e.,

$$m = \frac{y'}{x'}$$

The required equation is therefore

$$y - y' = \frac{y'}{x'}(x - x')$$

i.e.,

$$x'y - xy' = 0$$

Similary you can find the equation of normal to the general form of circle.

Note : The equation of normal to circle $x^2 + y^2 = a^2$ will always poss through centre and thus will be of the form $y = mx$

Illustration 27

Find the equation of the normal to the circle $x^2 + y^2 = 2x$, which is parallel to the line $x + 2y = 3$.

Solution :

Given circle is $x^2 + y^2 - 2x = 0$

Centre of given circle is (1, 0)

Since normal is parallel to $x + 2y = 3$

let the equation of normal is $x + 2y = \lambda$

Since normal passes through the centre of the circle i.e., (1, 0)

then $1 + 0 = l$

$\therefore$ $\lambda = 1$

then equation of normal is $x + 2y = 1$

or $x + 2y - 1 = 0$

Alternative Method :

Equation of normoal at (x_1, y_1) of $x^2 + y^2 - 2x = 0$ is

$$\frac{x - x_1}{x_1 - 1} = \frac{y - y_1}{y_1 - 0}$$

or slope $= \frac{y_1}{x_1 - 1} = m_1$ (say)

since normal is parallel to $x + 2y = 3$

$\therefore$ Slope $= -\frac{1}{2} = m_2$ (say)

but given $m_1 = m_2$

$$\frac{y_1}{x_1 - 1} = \frac{1}{2}$$

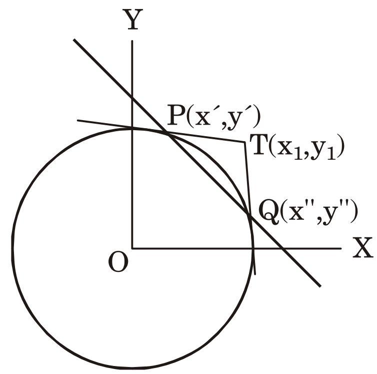
or $x_1 + 2y_1 - 1 = 0$

$\therefore$ locus of (x_1, y_1) is $x + 2y - 1 = 0$

13. Chord of Contact

If from any point T outside a circle two tangents TP and TQ be drawn to the circle, the straight line PQ joining the points of contact is called the chord of contact of tangents from T.

The equation of chord of contact of tangents to the circle at $x^2 + y^2 = a^2$ is given by



$$\boxed{xx_1 + yy_1 = a^2}$$

Proof :

Let T be the point (x_1, y_1) and P and Q the points (x´, y´) and (x´´, y´´) respectively.

The tangent at P is $\quad xx' + yy' = a^2 \quad$ ...(1)

and that at Q is, $\quad xx' + yy' = a^2 \quad$ ...(2)

Since, these tangents pass through T, its coordinates (x_1, y_1) must satisfy both (1) and (2).

Hence $\quad x_1x'' + y_1y'' = a^2 \quad$ ...(3)

and $\quad x_1x'' + y_1y'' = a^2 \quad$ ...(4)

The equation to PQ is then,

$$xx_1 + yy_1 = a^2 \quad ...(5)$$

For, since (3) is true, it follows that the point (x´, y´), i.e., P, lies on (5). Also, since (4) is true, it follows that the point (x´´, y´´), i.e., Q, lies on (5).

Hence, both P and Q lie on the straight line (5), i.e., (5), is the equation to the required chord of contact.

If the point (x_1, y_1) lie within the circle the argument of the preceding article will show that the line joining the (imaginary) points of contact of the two (imaginary) tangents drawn from (x_1, y_1) is $xx_1 + yy_1 = a^2$.

14. The equation of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ whose middle point is (x_1, y_1)

The equation of the chord o the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1y_1) is given by

$$\boxed{T = S_1}$$

where $\quad T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

and $\quad S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

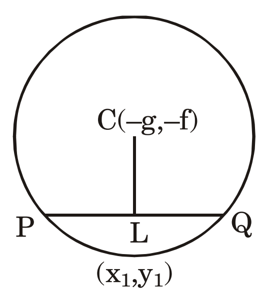
Proof :

The equation of given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

Its centre is $C \equiv (-g, -f)$

Let PQ be the chord whose mid-points is $L(x_1, y_1)$.

We have to find the equation of the chord PQ of the circle



$$\text{Slope of CL} = \frac{y_1 + f}{x_1 + g}$$

Since CL ⊥ PQ

$$\therefore \quad \text{slope of PQ} = \frac{x_1 + g}{y_1 + f}$$

Now equation of chord PQ is $y - y_1 = \frac{x_1 + g}{y_1 + f}(x - x_1)$

or $(y - y_1)(y_1 + f) = -(x_1 + g)(x - x_1)$

or $yy_1 - y_1^2 + yf - y_1f = -xx_1 + x_1^2 - xg + gx_1$

or $xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

Adding $gx_1 + fy_1 + c$ to both sides, we get

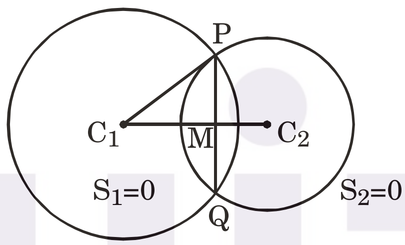
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

This is the required equation of the chord PQ whose mid-point is (x_1, y_1).

Note : This equation can be written as $T = S_1$,

15.1 Common chord of two Circles :

The chord joining the points of intersection of two given circles is called their common chord.



The equation of common chord of two circles is given by

$$\boxed{S_1 - S_2 = 0}$$

Where $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(i)

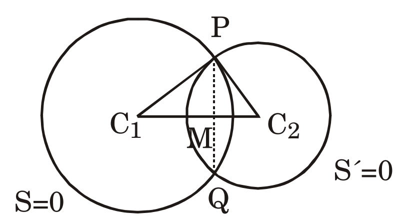
$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(ii)

subtracting (i) & (ii), we get $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

which is the required equation of common chord.

15.2 Length of common chord :

We have PQ = 2(PM) ($\because$ M is mid point of PQ)



$$= 2\sqrt{\{(C_1P)^2 - (C_1M)^2\}}$$

where C_1P = radius of the circle $S = 0$

and C_1M = length of perpendicular from C_1 on common chord PQ.

Illustration 28

Find the length of the common chord of the two circles

$x^2 + y^2 - 2x - 4y = 0$

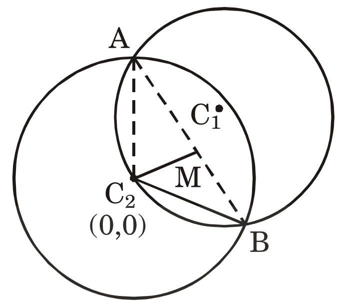
$x^2 + y^2 = 9$

Solution :

The equation of the common chord AB is $2x + 4y - 9 = 0$

Since centre of C_2 is (0, 0). The perpendicular length from C_2 to the common chord AB is

$$= \left|\frac{9}{\sqrt{2^2 + 4^2}}\right| = \frac{9}{\sqrt{20}}$$



$$\Rightarrow \quad AM = \sqrt{C_2A^2 - C_2M^2} = \sqrt{9 - \frac{81}{20}} = \frac{\sqrt{99}}{2\sqrt{5}}$$

$$\Rightarrow \quad \text{Length of common chord} = 2AM = \sqrt{\frac{99}{5}}$$

Illustration 29

Tangents are drawn at those points of the circle $x^2 + y^2 - 9 = 0$ where it is intersected by the circle $x^2 + y^2 - 2x - 4y = 0$. Find the intersection point of these tangents.

Solution :

Let the circles intersect at A and B and tangents to C_1 are drawn at A and B which intersect at P (α, β) then the line AB has two status. At first it is a common chord of two circles and secondly it is the chord of contact when tangents are drawn from P (α, β) to $x^2 + y^2 - 9 = 0$

$\Rightarrow$ The lines $2x + 4y - 9 = 0$ and $x\alpha + y\beta - 9 = 0$ are coincident.

$$\Rightarrow \quad \frac{\alpha}{2} = \frac{\beta}{4} = \frac{-9}{-9} \Rightarrow \alpha = 2, \beta = 4$$

Thus tangents intersect at (2, 4).

Illustration 30

Find the equation of the chord of $x^2 + y^2 - 6x + 10y - 9 = 0$ which is bisected at (– 2, 4).

Solution :

The equation of the required chord is

$$-2x + 4y - 3(x - 2) + 5(y + 4) - 9 = 4 + 16 + 12 + 40 - 9$$

$$-5x + 9y - 46 = 0$$

or

$$5x - 9y + 46 = 0$$

Illustration 31

Find the middle point of the chord intercepted on line $lx + my + n = 0$ by the circle $x^2 + y^2 = a^2$.

Solution :

Let (x_1, y_1) be the middle point of the chord intercepted by the circle $x^2 + y^2 = a^2$, on the line $lx + my + n = 0$. Then equation of the chord of the circle $x^2 + y^2 = a^2$, whose middle points is (x_1, y_1), is

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

or $$xx_1 + yy_1 = x_1^2 + y_1^2 \quad \ldots(1)$$

Clearly $lx + my + n = 0$ and (1) represented the same line,

$$\frac{x_1}{l} = \frac{y_1}{m} = \frac{x_1^2 + y_1^2}{-n} = \lambda \text{ (say)}$$

$\therefore$ $$x_1 = l\lambda, \quad y_1 = m\lambda \quad \ldots(2)$$

and $$x_1^2 + y_1^2 = -n\lambda$$

or $$l^2\lambda^2 + m^2\lambda^2 = -n\lambda \quad \text{(from (2))}$$

$\therefore$ $$\lambda = \frac{n}{l^2 + m^2}$$

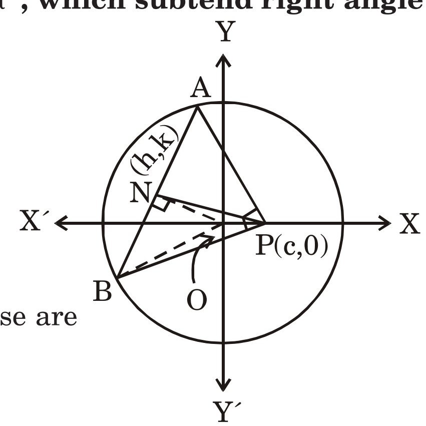
so from (2), $$x_1 = -\frac{nl}{l^2 + m^2}, \; y_1 = -\frac{mn}{l^2 + m^2}$$

Hence, the required point is $\left(-\frac{nl}{l^2 + m^2}, -\frac{nm}{l^2 + m^2}\right)$

Illustration 32

Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point $(c, 0)$.

Solution :



Let N (h, k) be the middle point of any chord AB, which subtend a right angle at P (c, 0)

Since $$\angle APB = 90$$

$\therefore$ $$NA = NB = NP$$

(since distance of the vertices from middle point of the hypotenuse are equal) or $(NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2$ ...(1)

But also $\angle BNO = 90$

$\therefore$ $(OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow$ $-(NB)^2 = (ON)^2 - (OB)^2$

$\Rightarrow$ $-[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

or $2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

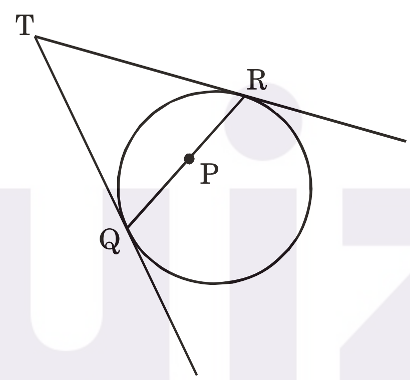
$\therefore$ Locus of N (h, k) is

$$2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$$

16. Pole and Polar

If through a point P (within or outside a circle) there be drawn any straight line to meet the circle in Q and R, the locus of the point of intersection of the tangents at Q and R is called the polar of P; also P is called the pole of the polar.

The equation to the polar of the point (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$.



$$\boxed{xx_1 + yy_1 = a^2}$$

In a similar manner it may be proved that the polar of (x_1, y_1) with respect to the circle.

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$\boxed{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$$

To find the pole of a given line with respect to any circle.

Let the equation to the given line be

$$Ax + By + C = 0 \quad ...(1)$$

1. Let the equation to the circle be,

$$x^2 + y^2 = a^2$$

and let the required pole be (x_1, y_1)

Then, (1) must be the equation to the polar of (x_1, y_1), i.e., it is the same as the equation.

$$xx_1 + yy_1 - a^2 = 0 \quad ...(2)$$

Comparing equations (1) and (2), we have

$$\frac{x_1}{A} = \frac{y_1}{B} = \frac{-a^2}{C}$$

so that $x_1 = \frac{A}{C}a^2$ and $y_1 = -\frac{B}{C}a^2$

The required pole is therefore, the point

$$\left(-\frac{A}{C}a^2, -\frac{B}{C}a^2\right)$$

2. Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

If (x_1, y_1) be the required pole, then (1) must be equivalent to the equation,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 \qquad ...(3)$$

i.e., $x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0$

Comparing (1) with (3), we therefore, have

$$\frac{x_1 + g}{A} = \frac{y_1 + f}{B} = \frac{gx_1 + fy_1 + c}{C}$$

By solving these equations we have the values of x_1 and y_1.

If the polar of a point P pass through a point T, then the polar of T passes through P.

Let P and T be the points (x_1, y_1) and (x_2, y_2) respectively.

The polar of (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$ is,

$$xx_1 + yy_1 = a^2$$

This straight line passes through the point T if,

$$x_2x_1 + y_2y_1 = a^2 \qquad ...(1)$$

Since the relation (1) is true, it follows that the point (x_1, y_1), i.e., P, lies on the straight line $xx_2 + yy_2 = a^2$, which is the polar of (x_2, y_2), i.e., T, with respect to the circle.

Problems based on pole and polar with respect to a circle

Working Rule : Use the following formula whichever is needed.

(i) In order to find the equation of the polar of point (x_1, y_1) in place of x and $\frac{x + y_1}{2}$ in place of y in the equation of the circle.

(ii) If the pole of the given line $lx + my + n = 0$ with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is to be obtained. Let the pole be (α, β) and then compare the equation of this polar and the equation of given line.

(iii) Two points are said to be conjugate to each other if the polar of one point w.r.t. the circle passes through the other.

Also two lines are conjugate of each other w.r.t. a circle if pole of one line lies on the other.

Illustration 33

Find the equation of the polar of the point (2, – 1) with respect to the circle $x^2 + y^2 - 3x + 4y - 8 = 0$.

Solution :

Given circle is $x^2 + y^2 - 3x + 4y - 8 = 0$ ...(1)

Given point is (2, – 1) let P ≡ (2, – 1). Now equation of the polar of point P with respect to circle (1) is

$$x.2 + y(-1) - 3\left(\frac{x+2}{2}\right) + 4\left(\frac{y-1}{2}\right) - 8 = 0$$

or $4x - 2y - 3x - 6 + 4y - 4 - 16 = 0$ or $x + 2y - 26 = 0$

Illustration 34

Find the pole of the line $3x + 5y + 17 = 0$ with respect to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$

Solution :

Given circle is $x^2 + y^2 + 4x + 6y + 9 = 0$ ...(1)

and given line is $3x + 5y + 17 = 0$ ...(2)

Let P (α, β) be the pole of line (2) with respect to circle (1).

Now equation of polar of point P(α, β) with respect to circle (1) is

$$x\alpha + y\beta + 2(x + \alpha) + 3(y + \beta) + 9 = 0$$

or $(\alpha + 2)x + (\beta + 3)y + 2\alpha + 3\beta + 9 = 0$ ...(3)

Now lines (2) and (3) are same, therefore,

$$\underset{\text{(i)}}{\frac{\alpha+2}{3}} = \underset{\text{(ii)}}{\frac{\beta+3}{5}} = \underset{\text{(iii)}}{\frac{2\alpha+3\beta+9}{17}}$$

From (i) and (ii), we get $5\alpha + 10 = 3\beta + 9$ or $5\alpha - 3\beta = -1$ ...(4)

From (i) and (iii), we get

$17\alpha + 34 = 6\alpha + 9\beta + 27$ or $11\alpha - 9\beta = -7$ ...(5)

Solving (4) and (5), we get α = 1, β = 2

Hence required pole is (1, 2).

Illustration 35

Show that the polars of the point (1, – 2) with respect to the circles $x^2 + y^2 + 2x + 8y + 5 = 0$ and $x^2 + y^2 + 2x + 8y + 5 = 0$ coincide. Prove also that there is another point, the polars of which with respect to these circles are the same and find its co-ordinates.

Solution :

Given circles are

$x^2 + y^2 + 6y + 5 = 0$ ...(1)

and $x^2 + y^2 + 2x + 8y + 5 = 0$ ...(2)

Let P ≡ (1, 2)

Now polar of point P (1, – 2) with respect to circle (1) is

$x.1 + y(-2) + 3(y - 2) + 5 = 0$ or $x + y - 1 = 0$ ...(3)

Again polar of point P (1, – 2) with respect to circle (2) is

$$x.1 + y(-2) + x + 1 + 4(y - 2) + 5 = 0$$

or $\quad 2x + 2y - 2 = 0 \quad \text{or} \quad x + y - 1 = 0 \quad \text{...(4)}$

From (3) and (4) it follows that the polars of point (1, – 2) with respect to circles (1) and (2) are same.

Second part :

Let $Q(\alpha, \beta)$ be a point the polars of which with respect to circles (1) and (2) are same.

Now equation of polars of point $P(\alpha, \beta)$ with respect to circles (1) and (2) are respectively.

$$x\alpha + y\beta + 3(y + \beta) + 5 = 0 \text{ or } \alpha x + (\beta + 3)y + 3\beta + 5 = 0 \quad \text{...(5)}$$

and $\quad x\alpha + y\beta + (x + \alpha) + 4(y + \beta) + 5 = 0$

or $\quad (\alpha + 1)x + (\beta + 4)y + \alpha + 4\beta + 5 = 0 \quad \text{...(6)}$

Now equations (5) and (6) are identical

$$\therefore \quad \underset{\text{(i)}}{\frac{\alpha + 1}{\alpha}} = \underset{\text{(ii)}}{\frac{\beta + 4}{\beta + 3}} = \underset{\text{(iii)}}{\frac{\alpha + 4\beta + 5}{3\beta + 5}}$$

From (i) and (ii), we have

$$\alpha\beta + \beta + 3\alpha + 3 = \alpha\beta + 4\alpha \quad \text{or} \quad \alpha - \beta = 3 \quad \text{...(7)}$$

From (i) and (iii), we have

$$3\alpha\beta + 3\beta + 5\alpha + 5 = \alpha^2 + 4\alpha\beta + 5\alpha$$

or $\quad \alpha^2 + \alpha\beta - 3\beta - 5 = 0$

or $\quad (\beta + 3)^2 + (\beta + 3)\beta - 3\beta - 5 = 0 \quad$ [(from (7)]

or $\quad 2\beta^2 + 6\beta + 4 = 0 \text{ or } \beta^2 + 3\beta + 2 = 0 \quad \therefore \beta = -1, -2.$

From (7), when $\beta = -1$, $\alpha = 2$ and when $\beta = -2$, $\alpha = 1$.

Thus points are (2, – 1) and (1, – 2).

But given P is (1, – 2), therefore, required point Q will be (2, – 1).

Illustration 36

Let C be the centre of a circle. The lines L_1 and L_2 are the polars of points A and B respectively with respect to the circle. Perpendiculars AM and BN are dropped from A to the line L_2 and from B to the line L_1. Prove that CA : CB = AM : BN.

Solution :

Let the circle be $\quad x^2 + y^2 - a^2 = 0 \quad \text{...(1)}$

$$A \equiv (x_1, y_1) \text{ and } B \equiv (x_2, y_2)$$

Since C is the centre of circle (1)

$$\therefore \; C \equiv (0, 0)$$

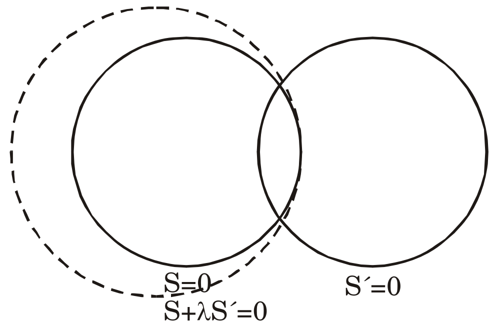
Now Polars of A and B w.r.t. circle (1) are respectively

$$xx_1 + yy_1 - a^2 = 0 \text{ ... (2)} \quad \text{and} \quad xx_2 + yy_2 - a^2 = 0 \quad \text{...(3)}$$

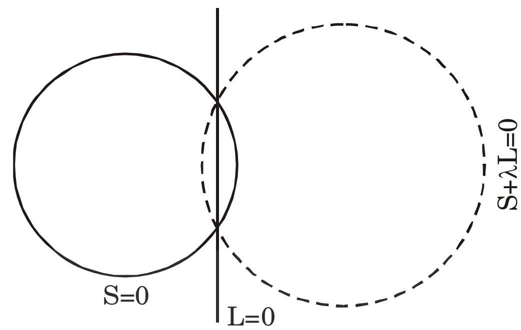
$$\therefore \quad \frac{AM}{BN} = \frac{|x_1x_2 + y_1y_2 - a^2|}{\sqrt{x_2^2 + y_2^2}} + \frac{|x_2x_1 + y_2y_1 - a^2|}{\sqrt{x_1^2 + y_1^2}} = \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} = \frac{CA}{CB}$$

17. Family of Circles

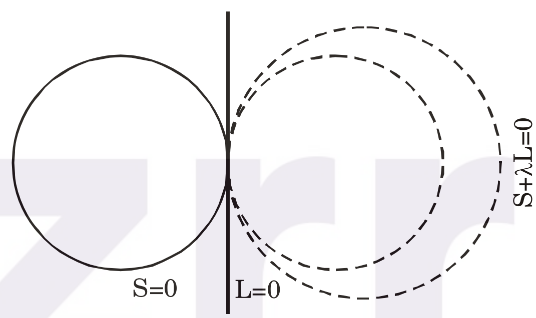
1. The equation of the family of circles passing through the point of intersection of two given circles S = 0 and S´ = 0 is given as $\mathbf{S + \lambda S' = 0}$ (where λ is a parameter, $\lambda \neq -1$)



2. The equation of the family of circles passing through the point of intersection of circle S = 0 and a line L = 0 is given as $\mathbf{S + \lambda L = 0}$ (where λ is a parameter)



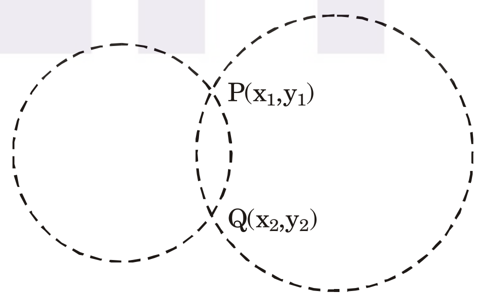
3. The equation of the family of circles touching the circle S = 0 and the line L = 0 at their point of contact P is $\mathbf{S + \lambda L = 0}$ (where λ is a parameter)



4. The equation of a family of circles passing through two given points $P(x_1, y_1)$ and (x_2, y_2) can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(where λ is a parameter)

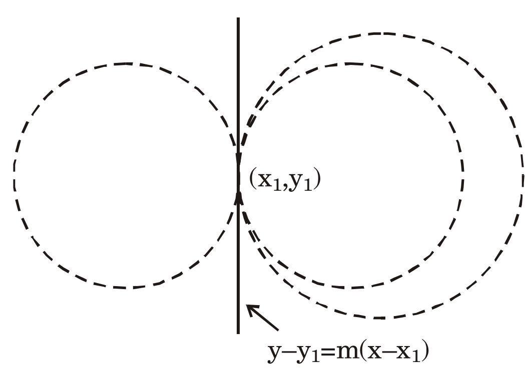


5. The equation of family of circles which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is $(x - x_1)^2 + (y - y_1)^2 + \lambda \{(y - y_1) - m(x - x_1)\} = 0$

and if m is infinite, the family of circles is

$(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$

(where λ is a parameter)



6. Equation of the circles given in diagram are
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$
$\pm \cot\theta \{(x - x_1)(y - y_2) - (x - x_2)(y - y_1)\} = 0$

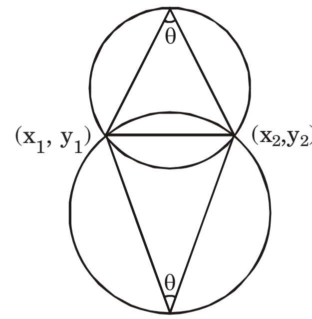


Illustration 37

Find the equation of a circle which passes through the intersection of the circles $x^2 + y^2 - 9 = 0$, $x^2 + y^2 + 2x + 4y + 3 = 0$ and also passes through (0, 0).

Solution :

The equation of the required circle must be of the form

$$x^2 + y^2 - 9 + \lambda(x^2 + y^2 - 2x - 4y + 3) = 0$$

If this passes through (0, 0) we must have

$$-9 + 3\lambda = 0 \Rightarrow \lambda = 3$$

The required circle, is $x^2 + y^2 - 9 + 3(x^2 + y^2 - 2x - 4y + 3) = 0$

or $$2x^2 + 2y^2 - 3x - 6y = 0$$

Illustration 38

Find the equation of a circle which passes through the intersection of the circles $x^2 + y^2 - 2x + 4y - 3 = 0$, $x^2 + y^2 - 6x - 8y + 5 = 0$ and

(i) whose centre lies on y-axis

(ii) whose centre lies on the line $2x + y = 7$

(iii) whose diameter is the common chord of given circles.

Solution :

The required circles must be of the form

$$x^2 + y^2 - 2x + 4y - 3 + \lambda(x^2 + y^2 - 6x - 8y + 5) = 0 \qquad ...(1)$$

which, on arranging becomes

$$(1 + \lambda)x^2 + (1 + \lambda)^2 - 2(1 + 3\lambda)x + 2(2 - 4\lambda)y + 5\lambda - 3 = 0$$

or $$x^2 + y^2 - 2.\frac{1+3\lambda}{1+\lambda}x + 2.\frac{2-4\lambda}{1+\lambda}y + \frac{5\lambda-3}{1+\lambda} = 0$$

The centre = $\left(\frac{1+3\lambda}{1+\lambda}, \frac{4\lambda-2}{1+\lambda}\right)$

For part (i), we must have $\frac{1+3\lambda}{1+\lambda} = 0$

$\Rightarrow \quad \lambda = -\frac{1}{3}$. On putting $\Rightarrow \lambda = -\frac{1}{3}$ at (1)

We get the answer to part (i)

For part (ii) we must have

$$2\left(\frac{1+3\lambda}{1+\lambda}\right)+\left(\frac{4\lambda-2}{1+\lambda}\right)=7 \Rightarrow \lambda=\frac{7}{3}$$

(iii) The common chord of the two given circle is $4x + 12y - 8 = 0$ or $x + 3y - 2 = 0$

Now the centre $\left(\frac{1+3\lambda}{1+\lambda}, \frac{4\lambda-2}{1+\lambda}\right)$ must lie on this common chord since common chord is the diameter

$$\Rightarrow \frac{1+3\lambda}{1+\lambda}+3\left(\frac{4\lambda-2}{1+\lambda}\right)-2=0 \qquad \Rightarrow \qquad \lambda=\frac{7}{13}$$

Note : The student will determine easily that certain problems of this type will either lead to indeterminate equations or equations which have no solutions. Let us discuss one case. Suppose we want the equation of circles passing through the intersection of the circles $x^2 + y^2 - 9 = 0$ and $x^2 + y^2 - 2x - 4y = 0$and passing through origin then on putting $x = 0$, $y = 0$ in

$$x^2 + y^2 - 9 + \lambda (x^2 + y^2 - 2x - 4y) = 0 \qquad ...(1)$$

we get $-9 = 0 \Rightarrow$ does not exist.

But it can easily noticed that the second circle satisfies both the condition. (It passes through origin and passes through the intersection of two given circles). Can we settle this paradox ? Indeed the equation $x^2 + y^2 - 2x - 4y = 0$ can be obtained from (1) if $\lambda = \infty$ or else, from the beginning itself we can take the family of the circles equation as

$$x^2 + y^2 - 2x - 4y + \lambda (x^2 + y^2 - 9) = 0$$

which will yield $\lambda = 0$

Thus in all such complex situations we must re-do the problem and must observe intricacies of the situation.

Illustration 39

Find the equation of a circle passing through the points (2, 0), (3, – 1) and (2, 5).

Solution :

The equation of circles passing through first two points (2, 0) and (3, – 1) may be taken as

$$(x-2)(x-3)+(y-0)(y+1)+\lambda\begin{vmatrix} x & y & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{vmatrix}=0$$

Or $\quad x^2 + y^2 - 5x + y + 6 + \lambda(x + y - 2) = 0$

If this passes through (2, 5) we must have

$$4 + 25 - 10 + 5 + 6 + 5\lambda = 0 \Rightarrow \lambda = -6$$

Thus the required circle is $x^2 + y^2 - 5x + y + 6 - 6(x + y - 2) = 0$

$\Rightarrow \quad x^2 + y^2 - 11x - 5y + 18 = 0$

Illustration 40

Find the equation of circle passing through (2, 0) and (3, – 1) and cutting a chord length 4 units on y-axis.

Solution :

From the equation $x^2 + y^2 - 5x + y + 6 + \lambda(x + y - 2) = 0$ we conclude that length of the chord intercepted on y-axis = $2\sqrt{\left(\frac{1+\lambda}{2}\right)^2 - (6-2\lambda)}$ (applying $2\sqrt{f^2 - c}$)

$$\Rightarrow \quad 2\sqrt{\left(\frac{1+\lambda}{2}\right)^2 - (6-2\lambda)} = 4$$

$\Rightarrow \quad \lambda = 3, \lambda = -13$

Thus one such circle is $x^2 + y^2 - 5x + y + 6 + 3(x + y - 2) = 0$

$\Rightarrow \quad x^2 + y^2 - 2x + 4y = 0$

Illustration 41

Show that any chord that arises as an intersection of a circle through A (2, 0) and B (3, – 1) and the circle $x^2 + y^2 - 2x + 6y - 11 = 0$ passes through a fixed point. Find the co-ordinates of that point.

Solution :

Any circle through A (2, 0) and B (3, – 1) is $x^2 + y^2 - 5x + y + 6 + \lambda(x + y - 2) = 0$

This will cut the circle $x^2 + y^2 - 2x + 6y - 11 = 0$ in a chord whose equation is

$$-3x - 5y + 17 + \lambda(x + y - 2) = 0$$

It is clear that these chords are concurrent at a point whose co-ordinates are given by

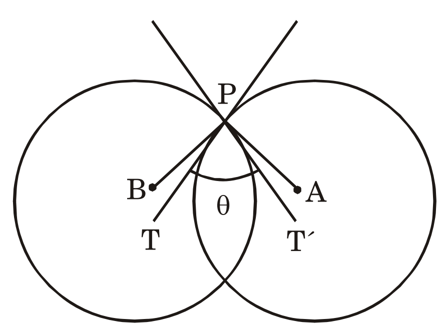
$$-3x - 5y + 17 = 0, \; x + y - 2 = 0$$

We easily get the fixed point as $\left(-\frac{7}{2}, \frac{11}{2}\right)$

18. Angle of Intersection of Two Circles

The angle between two circles is the angle between their tangents at their point of intersection.

Let P be a point of intersection of the two circles. Let PT and PT′ be the tangents to the two circles at P. Then the angle of intersection of the two circles at P is the ∠TPT′ or π – ∠TPT′

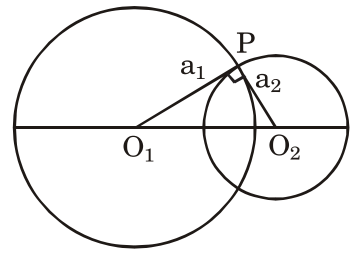


If A and B be the centres of the two circles then

PA ⊥ PT and PB ⊥ P′ ∴ ∠TPT′ = ∠APB or π – ∠APB.

19. ORTHOGONAL CIRCLES

Two circles are said to be intersect orthogonally when the tangents at their points of intersection are at right angles.



If the two circles intersect at P, the radii, O_1P and O_2P, which are perpendicular to the tangents at P, must also be at right angles.

Hence, $$O_1O_2^2 = O_1P^2 + O_2P^2$$

i.e., the square of the distance between the centres must be equal to the sum of the squares of the radii.

Also the tangent frrom O_2 to the other circle is equal to the radius a^2, i.e., if two circles be orthogonal the length of the tangent drawn from the centre of one circle to the second circle is equal to the radius of the first.

Either of these two conditions will determine whether the circles are orthogonal.

The centres of the circles,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ and } x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

are the points $(-g, -f)$ and $(-g', -f')$; also the squares of their radii are

$$g^2 + f^2 - c \text{ and } g'^2 + f'^2 - c'.$$

They therefore, cut orthogonally if

$$(-g + g')^2 + (-f + f')^2 = g^2 + f^2 - c + g'^2 + f'^2 - c',$$

i.e., if $$\boxed{2gg' + 2ff' = c + c'}$$

Illustration 42

Show that the circles $x^2 + y^2 - 2x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 6 = 0$ cut each other orthogonally.

Solution :

Given circles are $$x^2 + y^2 - 2x - 6y - 12 = 0 \quad ...(1)$$

and $$x^2 + y^2 + 6x + 4y - 6 = 0 \quad ...(2)$$

We know that the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

will cut each other orthogonally if $2gg_1 + 2ff_1 = c + c_1$

Here $$g = -1, f = -3, c = -12$$

and $$g_1 = 3, f_1 = 2, c_1 = -6$$

Now $$2gg_1 + 2ff_1 = 2(-1)(3) + 2(-3).2 = -18 = c + c_1$$

Hence circles (1) and (2) cut each other orthogonally

Illustration 43

If $S = 0$ and $S_1 = 0$ are the two circles with radii a and a_1 respectively. Show that the circles $\frac{S}{a} \pm \frac{S_1}{a_1}$ intersect at right angles.

Solution :

Let A and B be the centre of the two circles. We take AB as x-axis and its middle point as the origin. Let AB = 2k, then $A \equiv (-k, 0)$, $B \equiv (k, 0)$.

Now the equation of the two circles are

$$S \equiv (x - k)^2 + y^2 - a^2 = 0 \text{ and } S_1 \equiv (x + k)^2 + y^2 - a_1^2 = 0$$

$$\text{Also } \frac{S}{a} \pm \frac{S_1}{a_1} = 0 \quad \Rightarrow \quad a_1 S \pm a S_1 = 0$$

$$\Rightarrow \begin{cases} a_1[(x-k)^2 + y^2 - a^2] + a[(x+k)^2 + y^2 - a_1^2] = 0 \\ a_1[(x-k)^2 + y^2 - a^2] - a[(x+k)^2 + y^2 - a_1^2] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (a + a_1)(x^2 + y^2) - 2kx + (a_1 - a) + (k^2 - aa_1)(a_1 + a) = 0 \\ (a_1 - a)(x^2 + y^2) - 2kx(a_1 + a) + (k^2 + aa_1)(a_1 - a) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 + y^2 - 2k\left(\frac{a_1 - a}{a_1 + a}\right)x + k^2 - aa_1 = 0 & \text{...(1)} \\ x^2 + y^2 - 2k\left(\frac{a_1 + a}{a_1 - a}\right)x + k^2 + aa_1 = 0 & \text{...(2)} \end{cases}$$

Here $g = -\left(\frac{a_1 - a}{a_1 + a}\right)k,\ f_1 = 0,\ c = k^2 - aa_1$

and $g_1 = -\left(\frac{a_1 + a}{a_1 - a}\right)k,\ f_2 = 0,\ c_1 = k^2 + aa_1$

Now $2gg_1 + 2ff_1 = 2k^2 + 0 = 2k^2 = c + c_1$

Hence circles $\frac{S}{a} + \frac{S_1}{a_1} = 0$ and $\frac{S}{a} - \frac{S_1}{a_1} = 0$ intersect at right angles.

Illustration 44

Prove that the two circles which pass through the points (0, a) and (0, – a) and touch the line $y = mx + c$ will cut orthogonally if $c^2 = a^2(2 + m^2)$.

Solution :

Let a circle be $x^2 + y^2 + 2gx + 2fy + \lambda = 0$ ...(1)

Since circle (1) passes through (0, a) and (0, – a).

$\therefore$ $a^2 + 2fa + \lambda = 0$ and $a^2 - 2fa + \lambda = 0$

Solving these equations, we get $f = 0, \lambda = -a^2$

Since line $mx - y + c = 0$ touches circle (1)

$\therefore$ radius of the circle = length of the perpendicular from the centre (– g, – f) to the line.

$$\therefore \quad \sqrt{g^2 + f^2 - \lambda} = \frac{|-mg + f + c|}{\sqrt{1 + m^2}}$$

$$\therefore \quad \sqrt{g^2 + a^2} = \frac{|-mg + c|}{\sqrt{1 + m^2}} \quad [\because f = 0 \text{ and } \lambda = -a^2]$$

or $(1 + m^2)(g^2 + a^2) = (-mg + c)^2$

or $g^2 + 2mcg + (1 + m^2)a^2 - c^2 = 0$ ...(2)

Let g_1 and g_2 be the roots of eqn. (2)

$\therefore$ $g_1 + g_2 = -2mc, \; g_1g_2 = (1 + m^2)a^2 - c^2$

Corresponding to the two values of g the two circles are

$$x^2 + y^2 + 2g_1x - a^2 = 0 \text{ and } x^2 + y^2 + 2g_2x - a^2 = 0$$

They will cut orthogonally if $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

i.e. if, $2\{(1 + m^2)a^2 - c^2\} + 2.0.0 = -a^2 - a^2$

or $(1 + m^2)a^2 - c^2 + a^2 = 0$; or $c^2 = (2 + m^2)a^2$.

Illustration 45

Find the angle between the circles

$S : x^2 + y^2 - 4x + 6y + 11 = 0$ and $S' : x^2 + y^2 - 2x + 8y + 13 = 0$

Solution :

Centres and radii of circles S and S′ are

$$C_1(2, -3), \; r_1 = \sqrt{2}, \; C_2(1, -4), \; r_2 = 2$$

Distance between centres, $d = |C_1C_2|$

$$= \sqrt{(2-1)^2 + (-3+4)^2} = \sqrt{2}$$

If angle between the circles is θ, then

$$\cos(180 - \theta) = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$$

$$\cos(180 - \theta) = \left(\frac{2+4-2}{2.\sqrt{2}.2}\right) = \frac{1}{\sqrt{2}}$$

$$180 - \theta = 45$$

$$\theta = 135$$

20. RADICAL AXIS

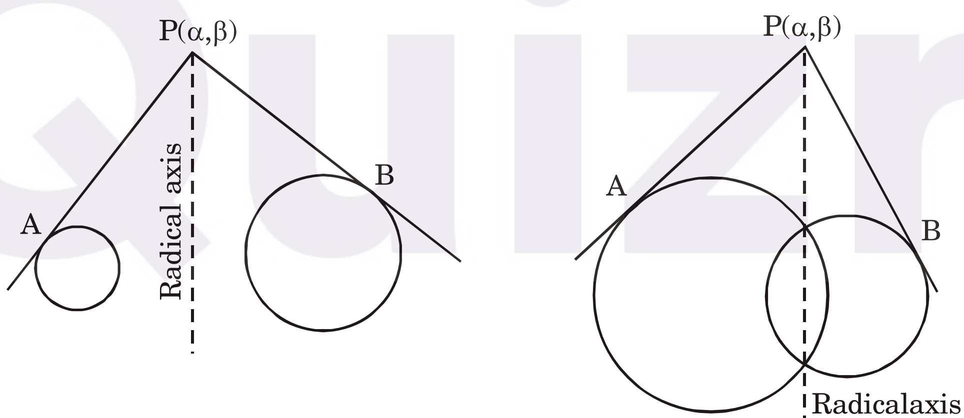
The radical axis of two circles is the locus of a point, which moves so that the lengths of the tangents drawn from it to the two circles are equal.

Let the equations of the two circles be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad ...(1)$$

and $$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad ...(2)$$

and let P(α, β) be any point such that the tangents from it to these circles are equal.



Now $PA = \sqrt{\alpha^2 + \beta^2 + 2g_1\alpha + 2f_1\beta + c_1}$

and $PB = \sqrt{\alpha^2 + \beta^2 + 2g_2\alpha + 2f_2\beta + c_2}$

We have PA = PB ∴ $PA^2 = PB^2$

∴ $\alpha^2 + \beta^2 + 2g_1\alpha + 2f_1\beta + c_1 = \alpha^2 + \beta^2 + 2g_2\alpha + 2f_2\beta + c_2$

or $2(g_1 - g_2)\,\alpha + 2\,(f_1 - f_2)\,\beta + c_1 - c_2 = 0$

Hence, the locus of P (α, β) is $2(g_1 - g_2)\,x + 2\,(f_1 - f_2)\,y + c_1 - c_2 = 0$

∴ The equation of the radical axis of circles (1) and (2) is

$$2(g_1 - g_2)\,x + 2\,(f_1 - f_2)\,y + c_1 - c_2 = 0$$

Note :

(i) If we denote the L.H.S. of (1) and (2) by S_1 and S_2 respectively, we see that the equation of the radical axis of the circles $S_1 = 0$ and $S_2 = 0$ will be $S_1 - S_2 = 0$

Hence in order to write down the equation of the radical axis of two circles make the co-efficient of x^2 in the equation of the two circles unity and then subtract.

(ii) Since $S_1 - S_2$ is of the form $S_1 + kS_2 = 0$, where $k = -1$ hence radical axis of two circles will pass through the point of intersection of the two circles if they cut each other. Thus

(a) if the two circles cut each other at two different points, then their radical axis will be their common chord.

(b) if the two circles touch each other (internally or externally), then their radical axis will be the common tangent at the point of contact.

(c) According to the above definition radical axis of two intersecting circles will be their common chord but from any point on this chord lying inside the circles no tangent can be drawn to the circles and hence definition of radical axis as given above is not correct and must be modified.

Modified definition of radical axis :

If $S_1 = 0$ and $S_2 = 0$ be the equations of the two circles where coefficients of x^2 in S_1 and S_2 are equal, then their radical axis is the straight line $S_1 - S_2 = 0$.

Properties of Radical axis :

Property I : The radical axis of two circles is perpendicular to the line of centres.

Let the equation of the two circles be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{...(1)}$$

and $$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \text{...(2)}$$

Let A and B be the centres of the two circles (1) and (2) respectively, then

$$A \equiv (-g_1, -f_1) \text{ and } B \equiv (-g_2, -f_2)$$

slope of $$AB = \frac{f_2 - f_1}{g_2 - g_1} = m_1 \text{ (say)}$$

$$2(g_1 - g_2)\,x + 2(f_1 - f_2)\,y + c_1 - c_2 = 0 \quad \text{...(3)}$$

Its slope $= \dfrac{g_1 - g_2}{f_1 - f_2} = -\dfrac{g_2 - g_1}{f_2 - f_1} = m_2$ (say)

since $m_1m_2 = -1$, hence radical axis is perpendicular to the line joining the centres.

Property II : The radical axis of three circles taken two at a time are concurrent.

Let the equation of the three circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad ...(1)$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad ...(2)$$

$$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad ...(3)$$

Radical axis of circles (1) and (2), (2) and (3) and (3) and (1) are

$$S_1 - S_2 \equiv = 2(g_1 - g_2)\,x + 2(f_1 - f_2)\,y + c_1 - c_2 = 0 \quad ...(4)$$

$$S_2 - S_3 \equiv = 2(g_2 - g_3)\,x + 2(f_2 - f_3)\,y + c_2 - c_3 = 0 \quad ...(5)$$

and $S_3 - S_1 \equiv = 2(g_3 - g_1)\,x + 2(f_3 - f_1)\,y + c_3 - c_1 = 0$ ...(6)

Adding (4), (5) and (6) we see that L.H.S. is indentically zero, hence the three lines are concurrent.

Note : The point of intersection of these three radical axes is called the radical centre of the three circles.

Property III : The radical axis of two circles bisects their common tangent.

Let AB be one of the common tangents meeting the radical axis at P, then since P lies on the radical axis hence by definition of radical axis PA = PB.

Thus the radical axis bisects the common chord.

Property IV : If two circles cut a third circle orthogonally, then the radical axis of the two circles will pass through the centre of the third circle.

Or

The locus of the centre of a circle cutting two given circles orthogonally is the radical axis of the two circles.

Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

cut the circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad ...(2)$$

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(3)

orthogonally, then

$$2gg_1 + 2ff_1 = c + c_1 \; ... (4) \text{ and } 2gg_2 + 2ff_2 = c + c_2 \quad ...(5)$$

(4) – (5), we get $2(g_1 - g_2)\,g + 2(f_1 - f_2)\,f = c_1 - c_2$ ...(6)

centre of circle (1) is $(-g, -f)$. $\therefore$ from (6) locus of $(-g, -f)$ is

$$2(g_1 - g_2)\,(-x) + 2(f_1 - f_2)\,(-y) = c_1 - c_2$$

or $2(g_1 - g_2)\,x + 2(f_1 - f_2)\,y + c_1 - c_2 = 0$ ...(7)

Clearly (7) is the radical axis of circles (2) and (3).

Radical Centre :

The point of concurrence of the radical axes of three circles whose centre are non-collinener, taken in pair, is called the radical centre of circles

Properties of Radical Centre :

1. Co-ordinates of radical centre can be found by solving the equations
$S_1 = S_2 = S_3 = 0$

2. The radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

3. The radical centre of three given circles will be the centre of a fouroth circle which cuts all the three circles orthogonally and the radius of the fourth circle is the length of tangent drawn from radical centre of the three given circles to any of these circles.

Illustration 46

Find the radical centre of three circles described on the three sides $4x - 7y + 10 = 0$, $x + y - 5 = 0$ and $7x + 4y - 15 = 0$ of a triangle as diameters.

Solution :

Since the radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

$\therefore$ Radical centre = orthocentre

Given sides are

$$4x - 7y + 10 = 0 \quad ...(1)$$
$$x + y - 5 = 0 \quad ...(2)$$
$$7x + 4y - 15 = 0 \quad ...(3)$$

Since lines (1) and (3) are perpendiculars the point of intersection of (1) and (3) is (1, 2), the orthocentre of the triangle. Hence radical centre is (1, 2).

21. Coaxial circles

Definition : A system of circles is said to be coaxial when they have a common radical axis, i.e., when the radical axis of each pair of circles of the system is the same.

Some properties

1. Circles passing through two fixed points form a coaxial system of circles, because every pair of circles has the same common chord and hence the same radical axis.
2. The equation $x^2 + y^2 + 2gx + c$, where g is variable and c is a constant, is the simplest equation of a coaxial system of circles. The common radical axis of this system of circles is y-axis.

To find the equation of a system of coaxial circles.

Since, the radical axis of any pair of the circle is perpendicular to the line joining their centres, it follows that the centres of all the circles of a coaxial system must lie on a straight line which is perpendicular to the radical axis.

Take the line of centres as the axis of x and the radical axis as the axis of y so that O is the origin.

The equation to any circle with its centre on the axis of x is

$$x^2 + y^2 - 2gx + c = 0 \qquad ...(1)$$

Any point on the radical axis is $(0, y_1)$.

The square on the tangent from it to the circle (1) is by $y_1^2 + c$.

Since, this quantity is to be the same for all circles of the system it follows that c is the same for all such circles; the different circles are therefore obtained by giving different values to g in the equation (1).

The intersections of equation (1) with the radical axis are then obtained by putting x = 0 in equation (1), and we have $y = \pm\sqrt{-c}$.

If c be negative, we have two real points of intersection. In such cases the circles are said to be of the Intersecting Species.

If c be positive, we have two imaginary points of intersection.

Illustration 47

Find the equation of the system of circles co-axial with the circles

$x^2 + y^2 + 4x + 2y + 1 = 0$

and $x^2 + y^2 - 2x + 6y - 6 = 0$

Also, find the equation of that particular circle whose centre lies on the radical axis.

Solution :

Given circles are

$$S_1 \equiv x^2 + y^2 + 4x + 2y + 1 = 0$$

and

$$S_2 \equiv x^2 + y^2 - 2x + 6y - 6 = 0$$

$\therefore$ Radical axis is

$$S_1 - S_2 = 0 \qquad ...(1)$$

i.e.,

$$6x - 4y + 7 = 0$$

Now system of co-axial circle is

$$S_1 + \lambda (S_1 - S_2) = 0$$

$$\Rightarrow \quad (x^2 + y^2 + 4x + 2y + 1) + \lambda (6x - 4y + 7) = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2x(2 + 3\lambda) + 2y(1 - 2\lambda) + 1 + 7\lambda = 0 \qquad ...(2)$$

Its central [– (2 + 3λ), – (1 – 2λ)] lies on (1)

$\therefore \quad 6 \quad -(2+3\lambda) - 4 \quad -(1-2\lambda) + 7 = 0$

or $\quad -12 - 18\lambda + 4 - 8\lambda + 7 = 0$

or $\quad -26\lambda - 1 = 0$

$$\therefore \quad \lambda = -\frac{1}{26}$$

Substituting the value of λ in (2), the equation of circle is

$$x^2 + y^2 + 2x\left(2 - \frac{3}{26}\right) + 2y\left(1 + \frac{2}{26}\right) + 1 - \frac{7}{26} = 0$$

$$\Rightarrow \quad 26(x^2 + y^2) + 98x + 56y + 9 = 0$$

Illustration 48

If A, B, C be the centres of three co-axial circles and t_1, t_2, t_3 be the lengths of the tangents to from any point, prove that $\overline{BC}.t_1^2 + \overline{CA}.t_2^2 + \overline{AB}t_3^2 = 0$

Solution :

Let the equations of three circles are $x^2 + y^2 + 2g_ix + c = 0$, i = 1, 2, 3,

According to the question A ≡ (– g_1, 0), B ≡ (– g_2, 0), C ≡ (– g_3, 0)

Let any point be P (h, k)

$$\therefore \quad t_1 = \sqrt{h^2 + k^2 + 2g_1h + c}$$

$$t_2 = \sqrt{h^2 + k^2 + 2g_2h + c}$$

$$t_3 = \sqrt{h^2 + k^2 + 2g_3h + c}$$

and $\quad \overline{AB} = (g_1 - g_2)$

$\overline{BC} = (g_2 - g_3)$

and $\quad \overline{CA} = (g_3 - g_1)$

Now $\overline{BC}.t_1^2 + \overline{CA}.t_2^2 + \overline{AB}t_3^2$

$= \Sigma (g_2 - g_3)(h^2 + k^2 + 2g_1h + c)$

$= (h^2 + k^2 + c)\,\Sigma (g_2 - g_3) + 2h\,\Sigma g_1 (g_2 - g_3)$

$= (h^2 + k^2 + c)(g_2 - g_3 + g_3 - g_1 + g_1 - g_2) + 2h \{g_1(g_2 - g_3) + g_2 (g_3 - g_1) + g_3 (g_1 - g_2)$

$= (h^2 + k^2 + c)(0) + 2h(0)$

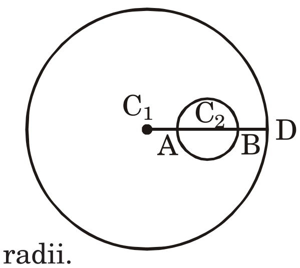
$= 0$

which proves the result.

22. The Common Tangents of Two circles

Let C_1 and C_2 be two circles whose radii are r_1 and r_2 respectively. Let $r_1 > r_2$ and let d be the distance between the centres C_1 and C_2. Five different cases arise.

Case 1 : C_2 completely within C_1



In this case $C_1D > C_1B$ (The line joining centres C_1 and C_2 cuts C_2 at A and B

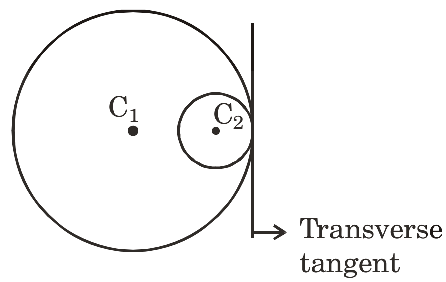
$\Rightarrow \quad r_1 > C_1C_2 + C_2B \Rightarrow r_1 > d + r_1 \Rightarrow d < r_1 - r_2$

condition : $d < |r_1 - r_2|$

No. of tangents : No tangent possible

The distance between the centres is less than the difference of the radii.

Case 2 : C_2 touches C_1 internally



condition : $d = r_1 - r_2$

No. of tangents : 1 common tangent

The distance between the centres is equal to the difference of their radii.

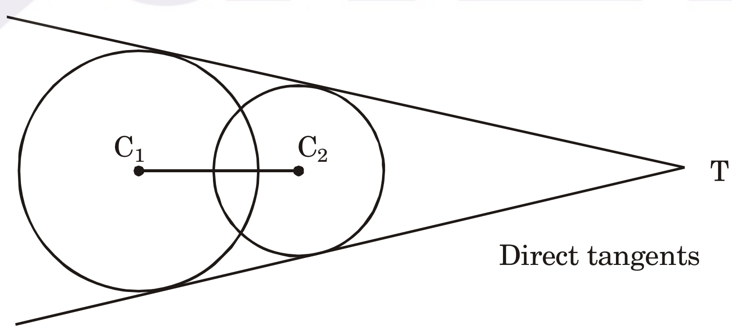
In this case there is one common tangent whose equation is $S - S' = 0$

Case 3 : C_1 & C_2 intersect (not just touching)

condition : $r_1 - r_2 < d < r_1 + r_2$

No. of tangents : 2 common tangents

The distance between the centres is greater than the difference of the radii and less than the sum of radii.



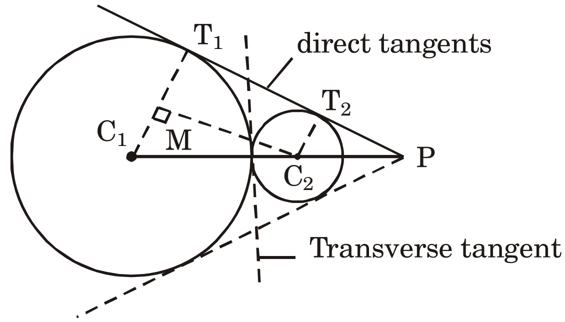
It is easy to observe by inspection that the necessary and sufficient condition for this case is $r_1 - r_2 < d < r_1 + r_2$

There will be two direct common tangents only. The length and the equation of tangents can be found as in the last case.

Case 4 : C_1 and C_2 touch externally

condition : $d = r_1 + r_2$

No. of tangents : 3 common tangents
1 transverse tangent
2 direct tangents



The length of direct common tangent = $T_1T_2 = C_2M = \sqrt{C_1C_2^2 - (C_1M)^2} = \sqrt{d^2 - (r_1 - r_2)^2}$

The equation of common tangent T_1T_2 can be obtained by finding the co-ordinate of the point P which is the point of intersection of T_1T_2 and C_1C_2 (produced). Note that

$$\frac{PC_1}{PC_2} = \frac{C_1T_1}{C_2T_2} = \frac{r_1}{r_2}$$

By external division formula P can be determined after which the common tangent T_1T_2 can be found.

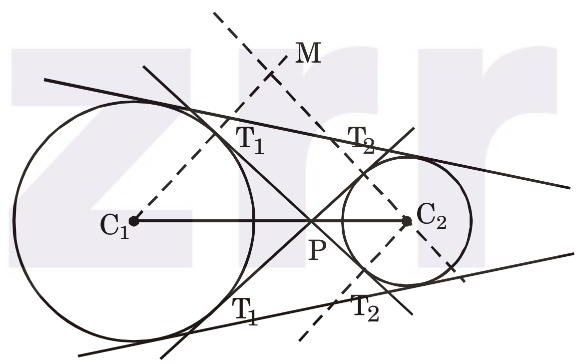
Case 5 : C_2 lies outside C_1 completely

condition : $d > r_1 + r_2$

No. of tangents : 4 common tangents
2 transverse tangents
2 direct tangents

The necessary and sufficient condition for this case is $d > r_1 + r_2$

The direct common tangents can be obtained by the method described in case (iii), the length of direct common tangent = $\sqrt{d^2 - (r_1 - r_2)^2}$



Now observe the following figure for length and equation of the transverse common tangents. Let one of the transverse common tangent meet C_1 at T_1 and C_2 at T_2 and T_1T_2 is intersected by C_1C_2 at P. Then $\frac{PC_1}{PC_2} = \frac{C_1T_1}{C_2T_2} = \frac{r_1}{r_2}$

from which P can be determined easily if C_1 and C_2 are given. The length $T_1T_2 = C_2M$ where M be the foot of the perpendicular from C_2 to C_1T_1 produced.

Now $C_2M = \sqrt{(C_1C_2)^2 - (C_1M)^2} = \sqrt{d^2 - (C_1T_1 + T_1M)^2} = \sqrt{d^2 - (r_1 + r_2)^2}$

Illustration 49

Determine the number of common tangents to the two circles
$C_1 : x^2 + y^2 = 25$, $C_2 : x^2 + y^2 - 4x + 6y + 4 = 0$ and find their lengths.

Solution :

The centres are (0, 0), (2, 3) and radii are 5 and 3.

$$d = \sqrt{13},\ r_1 = 5,\ r_2 = 3$$

We can note that $r_1 - r_2 < d < r_1 + r_2$

⇒ The circles intersect at two distinct real points.

⇒ There are two direct common tangents.

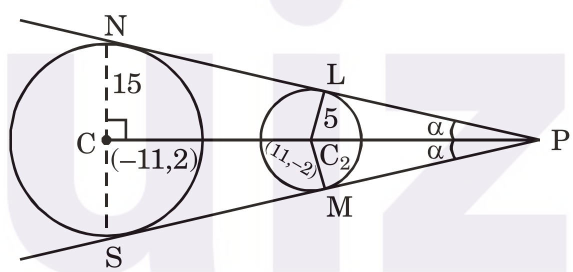
Their lengths $= \sqrt{d^2 - (r_1 - r_2)^2} = \sqrt{13 - (5-3)^2} = 3$ units

Illustration 50

Find all the common tangents to the circles $C_1 : x^2 + y^2 + 22x - 4y - 100 = 0$ and $C_2 : x^2 + y^2 - 22x + 4y + 100 = 0$. Find their lengths also.

Solution :

(C_1 and C_2 will also denote the centre of the two circles)



Note that centre of C_1 is (– 11, 2) and its radius is 15.

The centre of C_2 is (11, – 2) and its radius is 5.

Let the common tangent meet circle C_1 at N and C_2 at L.

Produce NL so that it intersects the line joining C_1 and C_2 at P.

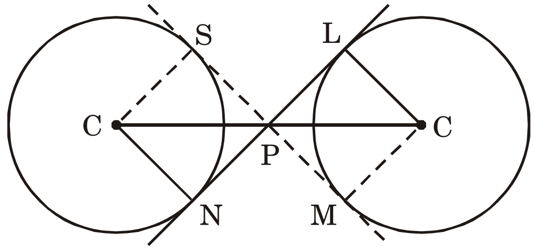
Since the triangle PNC_1 and PLC_2 are similar.

$$\frac{C_1P}{C_2P} = \frac{r_1}{r_2} = \frac{15}{3} = 3$$

Since P lies externally. The co-ordinates of P are

$$\left(\frac{15 \times 11 - 5(-11)}{15-5}, \frac{15 \times (-2) - 5 \times 2}{15-5}\right) \text{ i.e. P is } (22, -4)$$

Any line through (22, – 4) may be taken as $y + 4 = m(x - 22)$ or $y - mx + 4 + 22m = 0$



Since it is a tangent to C_2, we must have ⊥ from centre of C_2 = radius of C_2

$$\Rightarrow \quad \frac{-2-11m+4+22m}{\sqrt{1-m^2}} = \pm 5 \Rightarrow m = \frac{7}{24}, -\frac{3}{4}$$

⇒ The direct common tangents are $y+4 = \frac{7}{24}(x-22)$ and $y+4 = -\frac{3}{4}(x-22)$

or $7x - 24y = 250$ and $3x + 4y = 50$ ...(1)

Again for transverse common tangents

we note that common tangent NL is intersected by $C_1 C_2$ at P and $\frac{C_1P}{PC_2} = \frac{r_1}{r_2} = \frac{15}{5}$

$$\Rightarrow \quad P \text{ is } \left(\frac{15\times 11 + 5(-11)}{15+5}, \frac{15\times(-2)+5\times 2}{15+5}\right) \text{ i.e. P is } \left(\frac{11}{2}, -1\right)$$

Now any line through $\left(\frac{11}{2}, -1\right)$ may be taken as $y + 1 = m\left(x - \frac{11}{2}\right)$

As before we get $m = -\frac{24}{7}$ or $\frac{4}{3}$

⇒ Transverse common tangents are $y + 1 = -\frac{24}{7}\left(x - \frac{11}{2}\right)$ and $y + 1 = \frac{4}{3}\left(x - \frac{11}{2}\right)$

or $24x + 7y = 125$ or $4x - 3y = 25$ ...(2)

Note : If we take one of the common tangents (1) and solve it with C_1 and C_2. The distance between the points of contact will be the length of the direct common tangent. But the length can be found without finding these points and without finding the equation of common tangents. Indeed length of the direct

common tangents $= \sqrt{d^2 - (r_1 - r_2)^2} = \sqrt{(-11-11)^2 + (2+2)^2 - (15-5)^2} = \sqrt{400} = 20$

The length of the tranverse common tangent $= \sqrt{d^2 - (r_1 + r_2)^2} = 10$

Illustration 51

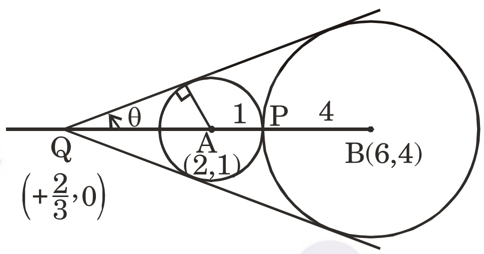
Find the coordinates of the point at which the circles $x^2 + y^2 - 4x - 2y + 4 = 0$ and $x^2 + y^2 - 12x - 8y + 36 = 0$ touch each other. Also find the equation of common tangents touching the circles in distinct points.

Solution :

I Circle	**II Circle**
Centre A(2, 1)	B (6, 4)
Radius $r_1 = \sqrt{4+1-4} = 1$	$r_2 = \sqrt{36+16-36} = 4$

$AB = \sqrt{16+9} = \sqrt{25} = 5 = 1 + 4 = r_1 + r_2$

$\therefore$ the two circles touch externally



P, the point of contact, divides AB internally in the ratio 1 : 4 hence P is $\left(\frac{14}{5}, \frac{8}{5}\right)$ and Q which divides AB externally in the same ratio is $\left(\frac{2}{3}, 0\right)$

The tangent from Q to the circle I will also be tangent to the circle II and hence will be a common tangent touching the two circles in two distinct points.

If θ be the angle that this tangent (actually there are two) makes with AB, then

$$\sin\theta = \frac{1}{AQ} = \frac{1}{\sqrt{1+\frac{16}{9}}} = \frac{3}{5}$$

$$\therefore \quad \tan\theta = \frac{3}{4}$$

Now if 'm' be the slope of one of the common tangents from Q, then

$$\frac{3}{4} = \tan\theta = \frac{m - \frac{3}{4}}{1 - m\frac{3}{4}} \left(\text{slope of AB} = \frac{3}{4}\right)$$

$$16\ m - 12 = 12 + 9\ m$$

$$7m = 24 \Rightarrow m = \frac{24}{7}$$

If 'm' be the slope of the other common tangent $\frac{m' - \frac{3}{4}}{1 + m'\frac{3}{4}} = \frac{-3}{4} \Rightarrow m' = 0$

The two common tangents are

$y - 0 = 0 \Rightarrow y = 0$

i.e., the x-axis

and $y - 0 = \frac{24}{7}\left(x - \frac{2}{3}\right)$

$7y = 24x - 16 \Rightarrow 24x - 7y - 16 = 0$

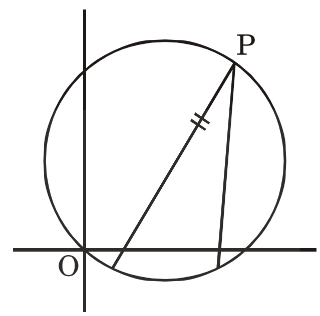
SOME MORE PROBLEMS

Illustration 1

Let a circle be given by 2x (x – a) + y (2y – b) = 0 ($a \neq 0$). Suppose it is possible to draw two distinct chords from $\left(a, \frac{b}{2}\right)$ on the circle such that each is bisected by x-axis. Show that the condition for this is $a^2 > 2b^2$.

Solution :

Let (α, β) be the other ends of the chord drawn from P $\left(a, \frac{b}{2}\right)$ then as it bisected by a point on x-axis we have



$$0 = \frac{\beta + \frac{b}{2}}{2} \Rightarrow \beta = -\frac{b}{2}$$

Since $\left(\alpha, -\frac{b}{2}\right)$ lies on the circle, we have

$$2\alpha(\alpha - a) - \frac{b}{2}(-2b) = 0 \text{ or } 2\alpha^2 - 2a\alpha + b^2 = 0$$

For two distinct such chords this should yield two distinct real roots.

$\Rightarrow \quad 4a^2 - gb^2 > 0$ or $a^2 > 2b^2$

Illustration 2

Show that the circle circumscribing the triangle whose sides are

x **cos** α **+** y **sin** α **=** a **sec**α **+** b **sin**α

x **cos** b **+** y **sin** β **=** a **sec**β **+** b **sin**β

x **cos**γ **+** y **sin**γ **=** a **sec**γ **+** b **sin**γ

passes through the points (0, b**)**

Solution :

The equation of a conic circumscribing the triangle must be of the form

$(x \cos\alpha + y \sin\alpha - a \sec\alpha - b \sin\alpha)(x \cos\beta + y \sin\beta - a \sec\beta - b \sin\beta)$

$+ \lambda (x \cos\beta + y \sin\beta - a \sec\beta - b \sin\beta)(x \cos\gamma + y \sin\gamma - a \sec\gamma - b \sin\gamma)$

$+ \mu (x \cos\gamma + y \sin\gamma - a \sec\gamma - b \sin\gamma)(x \cos\alpha + y \sin\alpha - a \sec\alpha - b \sin\alpha) = 0$ ...(i)

Since this represents a circle

coeff. of x^2 = coeff. of y^2 ...(ii)

coeff. of $xy = 0$ ...(iii)

(ii) $\Rightarrow$ $\cos\alpha\cos\beta + \lambda(\cos\beta\cos\gamma) + \mu(\cos\gamma\cos\alpha) = \sin\alpha\sin\beta + \lambda(\sin\beta\sin\gamma) + \mu(\sin\gamma\sin\alpha)$

$\Rightarrow$ $\cos(\alpha+\beta) + \lambda\sin(\beta+\gamma) + \mu\cos(\gamma+\alpha) = 0$ ...(iv)

(iii) $\Rightarrow$ $\sin(\alpha+\beta) + \lambda\sin(\beta+\gamma) + \mu\sin(\gamma+\alpha) = 0$ ...(v)

Solving (iv) and (v) for λ and μ, we get

$$\frac{\lambda}{\sin(\beta-\gamma)} = \frac{\mu}{\sin(\gamma-\alpha)} = \frac{1}{\sin(\alpha-\beta)}$$

$$\Rightarrow \quad \lambda = \frac{\sin(\beta-\gamma)}{\sin(\alpha-\beta)}, \mu = \frac{\sin(\gamma-\alpha)}{\sin(\alpha-\beta)}$$

Now the circle (i) will pass through (0, b) if

$(-a\sec\alpha)(-a\sec\beta) + \lambda(-a\sec\beta)(-a\sec\gamma) + \mu(-a\sec\gamma)(-a\sec\alpha) = 0$

Which is equivatent to $\cos\gamma + \lambda\cos\alpha + \mu\cos\beta = 0$ ...(vii)

To prove (vii) it is difficult to show $\cos\gamma + \dfrac{\sin(\beta-\gamma)}{\sin(\alpha-\beta)}\cos\alpha + \dfrac{\sin(\gamma-\alpha)}{\sin(\alpha-\beta)}\cos\beta = 0$

Which is indeed true in the light of C-D formulas.

Illustration 3

Find the range of parameter 'a' for which the variable line $y = 2x + a$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.

Solution :

The given circles are $C_1 : (x-1)^2 + (y-1)^2 = 1$ and $C_2 : (x-8)^2 + (y-1)^2 = 4$

The line $y - 2x - a = 0$ will lie between these circles if centre of the circles lie on opposite sides of the line, i.e., $(1-2-a)(1-16-a) < 0 \Rightarrow a \in (-15, -1)$

Line will not touch or intersect the circles if, $\dfrac{|1-2-a|}{\sqrt{5}} > 1, \dfrac{|1-16-a|}{\sqrt{5}} > 2$

$$\Rightarrow \quad |1+a| > \sqrt{5}, |15+a| > 2\sqrt{5}$$

$$\Rightarrow \quad a > \sqrt{5}-1 \text{ or } a < -\sqrt{5}, a > 2\sqrt{5}-15 \text{ or } a < -2\sqrt{5}-15$$

On taking the intersection of these inequations the required values of a lie in the interval

$$\left(2\sqrt{5}-15, -\sqrt{5}-1\right)$$

Illustration 4

Find the area of the quadrilateral formed by a pair of tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii.

Solution :

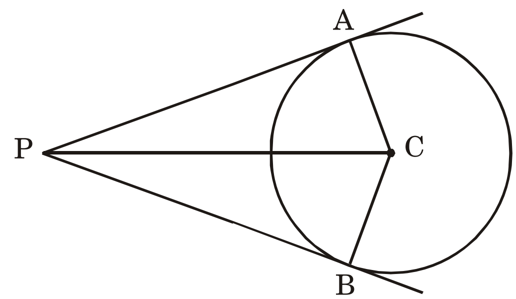
Given circle is $S \equiv x^2 + y^2 - 4x - 2y - 11 = 0$ ...(1)

Let C be its centre and a be its radius; then $C \equiv (2, 1)$ and $a = 4$.

Let $P \equiv (4, 5)$

Now length of tangent PA or PB from P to circle (1)

$$= \sqrt{4^2 + 5^2 - 4.4 - 2.5 - 11} = 2$$



and radius CA = 4

$\therefore$ area of ΔPAC

$$= \frac{1}{2}PA.AC = \frac{1}{2}2.4 = 4$$

$\therefore$ area of quadrilateral PACB

= 2 area of ΔPAC = 8 square units.

Illustration 5

Prove that the circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 23 = 0$

Solution :

Given circles are

$S_1 \equiv x^2 + y^2 - 6x - 4y + 9 = 0$ ...(1)

and $S_2 \equiv x^2 + y^2 - 8x - 6y + 23 = 0$ ...(2)

Equation of common chord of circles (1) and (2) which is also the radical axis of circles (1) and (2) is

$S_1 - S_2 = 0$

or $2x + 2y - 14 = 0$ or $x + y - 7 = 0$ ...(3)

Centre of circle (2) is (4, 3)

Clearly line (3) passes through the point (4, 3) and hence line (3) is the diameter of circle (2). Hence circle (1) bisects circumference of circle (2).

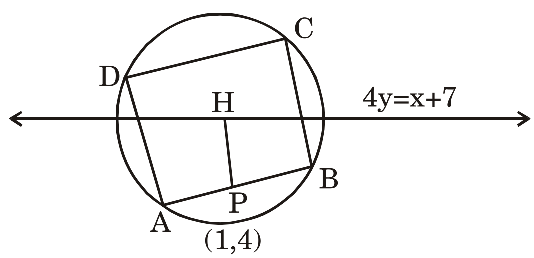
Illustration 6

One of the diameters of the circle circumscribing the rectangle ABCD is $4y = x + 7$. If A and B are the points (– 3, 4) and (5, 4) respectively, find the area of the rectangle.

Solution :

Given $A \equiv (-3, 4)$ and $B \equiv (5, 4)$

Let P be the middle point of AB, then $P \equiv (1, 4)$



Equation of AB is $y - 4 = \frac{4-4}{-3-5}(x+3)$ or $y = 4$, clearly

AB is parallel to x-axis, therefore perpendicular bisector of AB will be parallel to y-axis and since it passes through the point P (1, 4) therefore its equation will be x = 1. ...(1)

Also equation of one diameter of circle is 4y = x + 7 ...(2)

Solving (1) and (2), we get x = 1, y = 2

If H be the centre of the circle circumscribing the rectangle ABCD, then H ≡ (1, 2)

∴ length of perpendicular from H to AB i.e., $HP = \sqrt{(1-1)^2 + (4-2^2)} = 2$

∴ one side of the rectangle = 2HP = 4

Also other side of rectangle = AB $= \sqrt{(-3-5)^2 + (4+4)^2}$

Now area of rectangle ABCD = 4   8 = 32 units

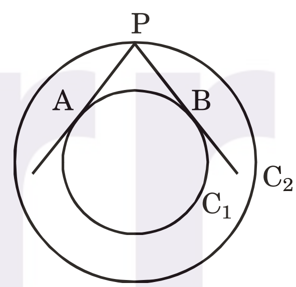
Illustration 7

C_1 and C_2 be two concentric circles. The radius of C_2 being twice that of C_1. From a point P on C_2 tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1.

Solution :

Let the equations of C_1 and C_2 be

$x^2 + y^2 = a^2$, $x^2 + y^2 = 4a^2$



From any point P (2a cos θ, 2a sin θ) tangents PA and PB are drawn then AB will be chord of contact whose equation must be

x. 2a cos θ + y. 2a sin θ = a^2 or x cos θ + y sin θ = a/2 ...(i)

If A be (x_1, y_1) and B be (x_2, y_2) must be roots of

$$x^2 + \left(\frac{\frac{a}{2} - x\cos\theta}{\sin\theta}\right)^2 = a^2$$ (since A, B be points of intersection of AB and C_1)

$$x^2 - a\cos\theta\, x + \left(\frac{a^2}{4} - a^2\sin^2\theta\right) = 0 \Rightarrow x_1 + x_2 = a\cos\theta$$

We similarly get $y_1 = y_2 = a \sin\theta$ (on eliminating x term)

Therefore the co-ordinates of centroid G of triangle PAB

$$= \left(\frac{2a\cos\theta + x_1 + x_2}{3}, \frac{2a\sin\theta + y_1 + y_2}{3}\right) = (a\cos\theta, a\sin\theta)$$

Which obviously lies on C_1.

Illustration 8

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of foot of the ⊥ drawn from origin upon any any chord of S = 0 which subtend a right angle at the origin.

Solution :

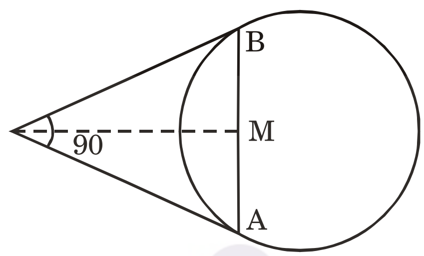
Let AB be any arbitrary chord subtending 90 at origin whose equation is

$$y = mx + d \qquad ...(i)$$

Then equation of OA and OB can be obtained by homogenizing equation of S with the help of (i)

Thus equation of OA and OB is

$$x^2 + y^2 + 2gx\left(\frac{y - mx}{d}\right) + 2fy\left(\frac{y - mx}{d}\right) + c\left(\frac{y - mx}{d}\right)^2 = 0 \qquad ...(ii)$$



Since OA ⊥ OB (given) in equation (ii), coeff. of x^2 + coeff. of $y^2 = 0$

$$\Rightarrow \qquad 1 - \frac{2gm}{d} + \frac{cm^2}{d^2} + 1 + \frac{2f}{d} + \frac{c}{d^2} = 0 \qquad ...(iii)$$

Now equation of line ⊥ to AB and passing through origin must be

$$y = -\frac{1}{m}x \qquad ...(iv)$$

Since the point M whose locus is to be determined is point of intersection of OM and AB the required locus must be eliminant of (i), (iv) and (iii). On solving (i) and (iv) for m and d,

we get $\qquad m = -\dfrac{x}{y}$ and $d = \dfrac{x^2 + y^2}{y}$

on putting m and d in (iii), we get

$$1 - 2g\left(\frac{-x}{x^2 + y^2}\right) + c\left(-\frac{x}{x^2 + y^2}\right)^2 + 1 + 2f\frac{y}{x^2 + y^2} + \frac{cy^2}{\left(x^2 + y^2\right)^2} = 0$$

which simplifies to $x^2 + y^2 + gx + fy + \dfrac{c}{2} = 0$ (on multiplying by $(x^2 + y^2)^2$).

Illustration 9

A circle touches the line y = x at P such that OP = $4\sqrt{2}$. The circle contains (– 10, 2) in its interior and the lengths of its chord on the line x + y = 0 is $6\sqrt{2}$. Determine the equation of the circle.

Solution :

Let y = x cuts circle at A and B and M is the mid point of AB. Then if C be the centre of the circle CPOM must be a rectangle.

($\angle CMO = 90$, $CPO = 90$, $(y = x) \perp (y = -x)$)

$\Rightarrow \quad CM = OP = 4\sqrt{2}$

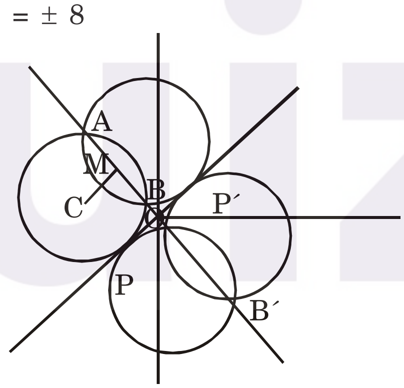
$\Rightarrow \quad$ radius $CA = \sqrt{CM^2 + MA^2} = \sqrt{(4\sqrt{2})^2 + (3\sqrt{2})^2} = 5\sqrt{2}$

If C be (h, k) then ⊥ from (h, k) on y = x must be equal to radius

$\Rightarrow \quad \dfrac{k-h}{\sqrt{2}} = \pm 5\sqrt{2} \Rightarrow k - h = \pm 10$ ...(i)

Again ⊥ from (h, k) y = – x must be equal to $4\sqrt{2}$

$\Rightarrow \quad \dfrac{k+h}{\sqrt{2}} = 4\sqrt{2} \Rightarrow k + h = \pm 8$ ...(ii)



In all, there are four possibilities in which equations (i) and (ii) can exist

k – h	k + h	Solution	
+ 10	+ 8	h = – 1, k = 9	The circle is $(x + 1)^2 + (y - 9)^2 = (5\sqrt{2})^2$ this doesn't contain (– 10, 2) in its interior
+ 10	– 8	h = – 9, k = 1	The circle is $(x + 9)^2 + (y + 1)^2 = (5\sqrt{2})^2$ and which satisfies all given conditions.
– 10	+ 8	h = 9, k = – 1	The circle does not contain (– 10, 2)
– 10	– 8	h = 1, k = – 9	The circle does not contain (– 10, 2)

Thus the only circle satisfying the condition of the problem is $(x + 9)^2 + (y + 1)^2 = (5\sqrt{2})^2$

Illustration 10

A ray is drawn from origin to cut the given circle $x^2 + y^2 = 2ax$ ($a > 0$) at B. From B equal segments BM and BN of constant length b are laid of in either direction. As the ray revolves the points M nd N describe a curve (limacon of Pascal). Find its equation.

Solution :

Let $y = mx$ be any ray OB drawn from O then for B

we solve $y = mx$ and $x^2 + y^2 - 2ax = 0$. We easily get $x = \frac{2a}{1+m^2}, \ y = \frac{2am}{1+m^2}$

Let us now put equation of OB in parametric from with respect to point B (since M and N are situated at a distance b from B).

Indeed equation of OB is $y - \frac{2am}{1+m^2} = m\left(x - \frac{2a}{1+m^2}\right)$

or $$\frac{x - \frac{2a}{1+m^2}}{1/\sqrt{1+m^2}} = \frac{y - \frac{2am}{1+m^2}}{m/\sqrt{1+m^2}}$$

For the points M and N we can equate to ± b and the squared result will be satisfied by both M and N. Taking first expression equal to b and on squaring, we get $\left(x - \frac{2a}{1+m^2}\right)^2 = \frac{b^2}{1+m^2}$

But M and N satisfy the simple relation $y = mx$ (since they lie on OB)

Therefore on putting $m = \frac{y}{x}$, we get a pure relation between abscissa and co-ordinates of M

(or N) as $$\left(x - \frac{2a}{1 + \frac{y^2}{x^2}}\right)^2 = \frac{b^2}{1 + \frac{y^2}{x^2}}$$

which easily simplifies to $(x^2 + y^2 - 2ax)^2 = b^2(x^2 + y^2)$ and which is the equation of limacon of Pascal.

Illustration 11

Consider two circles $C_1 : x^2 + y^2 = r_1^2$ and $C_2 : x^2 + y^2 = r_2^2$ ($r_2 > r_1$). Let 'A' be a fixed point on the circle C_1 say A $(r_1, 0)$ and 'B' a variable point on the circle C_2. The line BA meets the circle C_2 again at C. Find

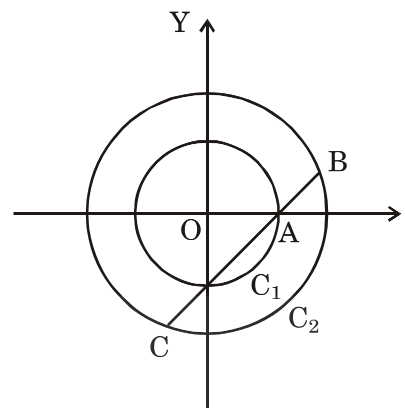
(a) show that $OA^2 + OB^2 + BC^2 \in [5{r_2}^2 - 3{r_1}^2, 5{r_2}^2 + {r_1}^2]$

(b) the locus of mind point of AB, 'O' being the origin

Solution :

A must be $(r_1, 0)$. Let B be the point $(r_2 \cos\theta, r_2 \sin\theta)$, it is evident that the length of BC will be maximum when BC is the diameter of C_2 and minimum when BC is tangent to C_1 at A.

$\Rightarrow$ max BC = $2r_2$, min BC = $2\sqrt{r_2^2 - r_1^2}$



Now, $OA^2 + OB^2 + BC^2 = r_1^2 + r_2^2 + BC^2$, which will lie between

$$r_2^2 + r_1^2 + 4(r_2^2 - r_1^2) \text{ and } r_2^2 + r_1^2 + 4r_2^2$$

$$\Rightarrow \quad 5r_2^2 + r_1^2 \geq OA^2 + OB^2 + BC^2 \geq 5r_2^2 - 3r_1^2$$

$$OA^2 + OB^2 + BC^2 \in [5r_2^2 - 3r_1^2, 5r_2^2 + r_1^2]$$

Now let 'D' be the mid-point of AB

$$\Rightarrow \quad D \equiv \left(\frac{r_1 + r_2\cos\theta}{2}, \frac{r_2\sin\theta}{2}\right) = (h, k) \quad \text{(say)}$$

$$\Rightarrow \quad \sin\theta = \frac{2k}{r_2}, \cos\theta = \frac{2h - r_1}{r_2} \Rightarrow 4k^2 + (2h - r_1)^2 = r_2^2$$

Locus of 'D' is, $\left(x - \frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{4}$

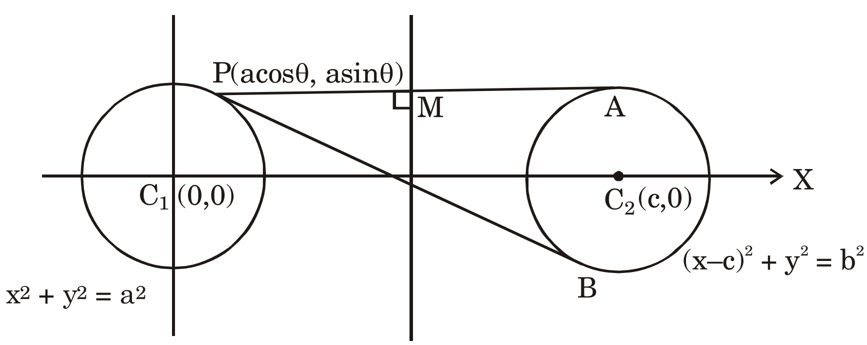
Illustration 12

Prove that the square of the length of the tangent drawn from a point on one circle to another circle is equal to twice the product of the perpendicular distance of that point from the radical axis of the two circles and the distance between their centres.

Solution :

Without loss of generality, the equation of the two circles can be taken as

$$x^2 + y^2 = a^2 \text{ and } (x - c)^2 + y^2 = b^2$$



$\therefore$ Radical axis of the two circles is given by $(x-c)^2 - x^2 - b^2 + a^2 = 0$ ...(1)

$$\Rightarrow \quad -2cx + c^2 + a^2 - b^2 = 0 \Rightarrow x = \frac{c^2+a^2-b^2}{2c}$$

Let P (a cos θ, a sin θ) be any point on the first circle, then

$$PA = \sqrt{(a\cos\theta - c)^2 + a^2\sin^2\theta - b^2} = \sqrt{a^2 - b^2 + c^2 - 2ac\cos\theta}$$

Also, $PM = \dfrac{c^2+a^2-b^2}{2c} - a\cos\theta = \dfrac{c^2+a^2-b^2-2ac\cos\theta}{2c}$ and $C_1C_2 = c$

$$\Rightarrow \quad 2(PM).(C_1C_2) = \frac{c^2+a^2-b^2-2ac\cos\theta}{c}.c = c^2+a^2-b^2-2ac\cos\theta = (PA)^2$$

Hence, $(PA)^2 = 2\,(PM).\,(C_1C_2)$

Illustration 13

Tangents are drawn from the point (h, k) to the circle $x^2 + y^2 = a^2$. Prove that the area of the triangle formed by them and their chord of contact is $\dfrac{a(h^2+k^2-a^2)^{\frac{3}{2}}}{h^2+k^2}$

Solution :

Given circle is $x^2 + y^2 = a^2$ ...(1)

The equation of the chord of contact AB of tangents drawn from P(h, k) to the circle (1) is

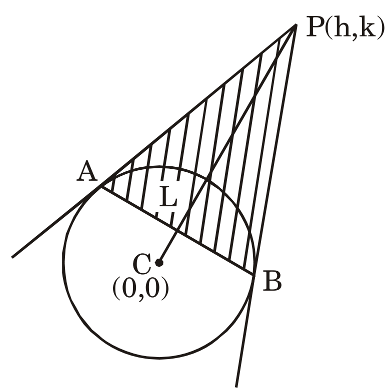
$xh + yk = a^2$ ...(2)

We have to find the area of ΔPAB. From P(h, k) draw PL ⊥ AB.

Now $PL = \dfrac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}$ ...(3)

Here $h^2 + k^2 - a^2 > 0$ $\because$ P(h, k) lies outside circle (1)

Also $PA = \sqrt{h^2+k^2-a^2}$ ...(4)



$$\therefore \quad AL^2 = AP^2 - PL^2 = (h^2+k^2-a^2) - \frac{(h^2+k^2-a^2)^2}{h^2+k^2}$$

$$= \frac{(h^2+k^2-a^2)(h^2+k^2) - (h^2+k^2-a^2)^2}{h^2+k^2}$$

$$\therefore \quad AL = \sqrt{\frac{(h^2+k^2-a^2)(h^2+k^2-h^2-k^2+a^2)}{(h^2+k^2)}}$$

$$= \frac{a(h^2 + k^2 - a^2)^{\frac{1}{2}}}{(h^2 + k^2)^{\frac{1}{2}}}$$

Now the area of $\Delta APB = \frac{1}{2}.AB.PL = AL.PL$

$$= \frac{a(h^2 + k^2 - a^2)^{\frac{1}{2}}}{(h^2 + k^2)^{\frac{1}{2}}} . \frac{(h^2 + k^2 - a^2)}{(h^2 + k^2)^{\frac{1}{2}}} = \frac{a(h^2 + k^2 - a^2)^{\frac{3}{2}}}{h^2 + k^2}$$

Illustration 14

Find the equation of a circle which is coaxial with the circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ and $x^2 + y^2 + 4x + 2y + 1 = 0$. It is given that the centre of the circle to be determined lies on the radical axis of these circles.

Solution :

Equation of the given circles are

$$S_1 \equiv x^2 + y^2 - x + 3y - \frac{3}{2} = 0 \quad ...(1)$$

$$S_2 \equiv x^2 + y^2 + 4x + 2y + 1 = 0 \quad ...(2)$$

$\therefore$ The radical axis of circles (1) and (2) is

$$S_1 - S_2 = O \text{ or } -5x + y - \frac{5}{2} = 0 \text{ or } 10x - 2y + 5 = 0 \quad ...(3)$$

$\therefore$ Required circle will have the equation of the form

$$x^2 + y^2 + 4x + 2y + 1 + k(10x - 2y + 5) = 0$$

or $$x^2 + y^2 + 2(2 + 5k)x + 2(1 - k)y + (1 + 5k) = 0 \quad ...(4)$$

Its centre is $(-2 - 5k, k - 1)$. From question it lies on line (3)

$\therefore$ $10(-2 - 5k) - 2(k - 1) + 5 = 0$; or $-52k - 13 = 0$;

$\therefore$ $k = -\frac{1}{4}$

Putting the value of k in (4), we get

$$x^2 + y^2 + 2\left(2 - \frac{5}{4}\right)x + 2\left(1 + \frac{1}{4}\right)y + \left(1 - \frac{5}{4}\right) = 0$$

or $$x^2 + y^2 + \frac{3}{2}x + \frac{5}{2}y - \frac{1}{4} = 0;$$

or $$4(x^2 + y^2) + 6x + 10y - 1 = 0 \quad ...(5)$$

This is the required circle.