



ENGINEERING PHYSICS

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The Nabla operator is a differential vector operator

$$\vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \quad \dots \text{Del operator}$$

$$\vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 \quad \dots \text{Laplacian operator}$$

$$\vec{\nabla} \times (\vec{\nabla} \times A) = \vec{\nabla}(\vec{\nabla} \cdot A) - \nabla^2 A \quad \dots \text{Vector identity}$$

Operations with the Nabla operator (del operator)

- $\vec{\nabla}$ operates on a scalar to give a vector
 - Gradient of the scalar
- The dot product (.) of $\vec{\nabla}$ with a vector gives a scalar
 - Divergence of the vector
- The cross product (\times) of $\vec{\nabla}$ with a vector gives a vector
 - Curl of the vector



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Gradient of a scalar field

Gradient of a scalar $V(xyz)$

$$\mathbf{grad} V = \nabla V = \hat{i} \frac{\partial V}{\partial x} + \hat{j} \frac{\partial V}{\partial y} + \hat{k} \frac{\partial V}{\partial z}$$

The gradient of a scalar field gives a vector

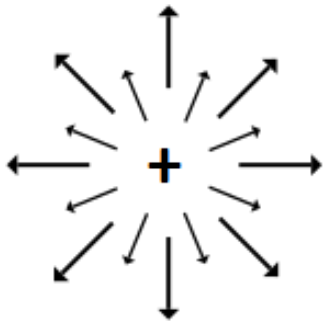
Gradient gives the rate of change of the property at any point and the direction gives the direction in which the change is maximum

Divergence of a vector field

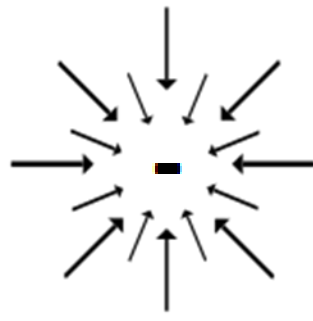
Divergence of a vector $\vec{V} = \hat{i}V_x + \hat{j}V_y + \hat{k}V_z$

$$\text{Div } V = \nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

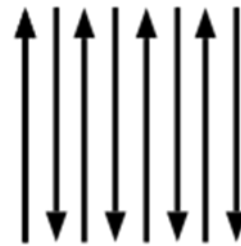
The divergence of a vector field gives a scalar



$$\begin{aligned}\frac{\partial}{\partial x}(V_x) &> 0 \\ \frac{\partial}{\partial y}(V_y) &> 0 \\ \nabla \cdot (\mathbf{V}) &> 0\end{aligned}$$



$$\begin{aligned}\frac{\partial}{\partial x}(V_x) &< 0 \\ \frac{\partial}{\partial y}(V_y) &< 0 \\ \nabla \cdot (\mathbf{V}) &< 0\end{aligned}$$

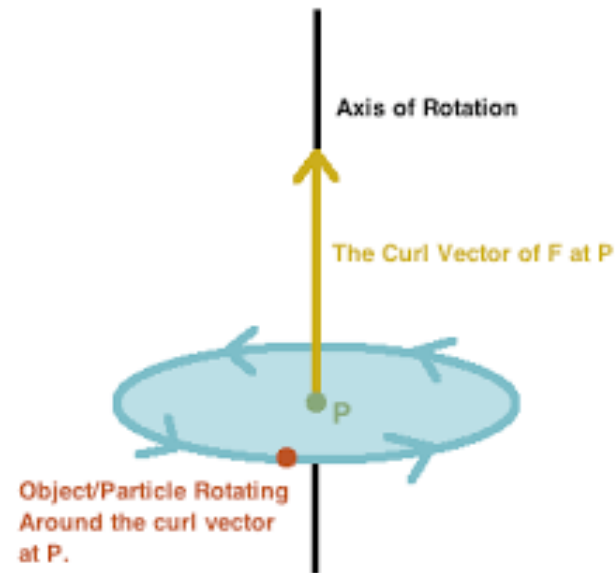
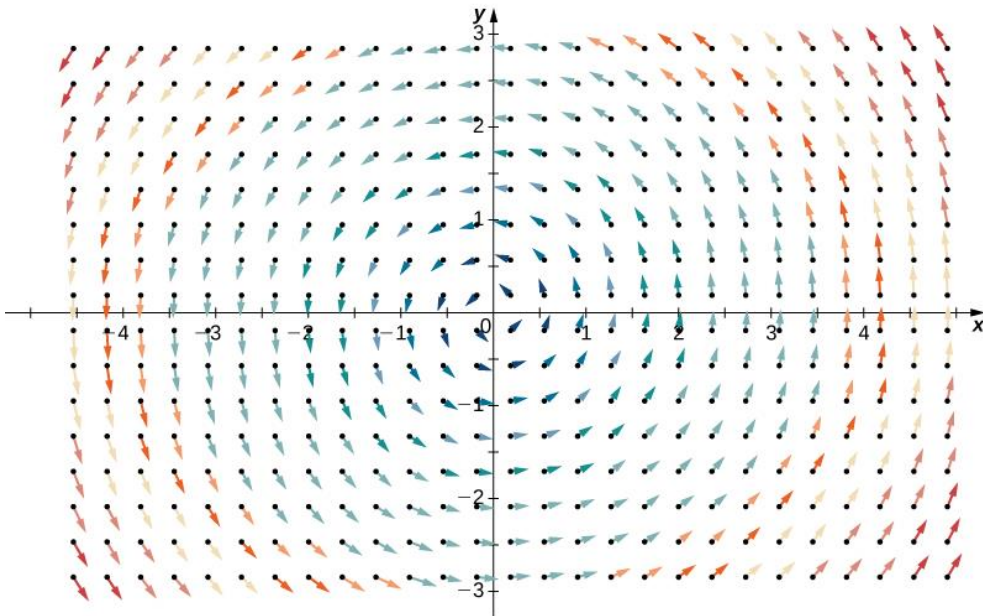


$$\begin{aligned}\frac{\partial}{\partial x}(V_x) &= 0 \\ \frac{\partial}{\partial y}(V_y) &= 0 \\ \nabla \cdot (\mathbf{V}) &= 0\end{aligned}$$

Curl of a vector field

$$\text{curl } A = \nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

The curl of a vector is another vector



Maxwell's Electric Flux equation:

Divergence of the electric field is given by the charge density divided by ϵ_0

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Maxwell's Magnetic Flux equation:

Divergence of the magnetic field is uniformly zero

$$\nabla \cdot \vec{B} = 0$$

This implies the absence of magnetic monopoles

Maxwell's equation for induced electric field:

The curl of the induced electric field in a closed loop is proportional to the rate of change of magnetic flux linked with the loop.

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

Maxwell's equation for induced magnetic field:

The curl of the magnetic field in a closed loop is equal to the sum of the current density and the displacement current due to the time varying electric field

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

1. Gauss' Law



$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

2. Gauss' Law for Magnetism



$$\nabla \cdot \vec{B} = 0$$

3. Faraday's Law



$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

4. Ampere-Maxwell Law



$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

A general wave equation,

$$\nabla^2 \vec{A} = \left(\frac{1}{v^2} \frac{\partial^2 \vec{A}}{\partial t^2} \right), \text{ with velocity} = v$$

Laplacian operator

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \mathbf{A}) = \vec{\nabla}(\vec{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Maxwell's equations in free space

Free space implies – charges and currents do not exist

$$\vec{\nabla} \cdot \vec{E} = 0 \quad 1.$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad 2.$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad 3$$

$$\vec{\nabla} \times \vec{B} = +\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad 4.$$

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Electric waves in free space



Taking the curl of Maxwell's equation 3

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(- \frac{\partial \vec{B}}{\partial t} \right)$$

this reduces to, $\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = \left(- \frac{\partial \vec{\nabla} \times \vec{B}}{\partial t} \right)$

For free space, $\vec{\nabla} \cdot \vec{E} = 0$ (Maxwell's equation 1),

$$\text{Thus, } -\nabla^2 \vec{E} = \left(- \frac{\partial \vec{\nabla} \times \vec{B}}{\partial t} \right)$$

Substituting for curl of B (Maxwell's equation 4)

$$\nabla^2 \vec{E} = \left(\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

with $\mu_0 \epsilon_0 = \frac{1}{c^2}$, wave equation for electric wave in free space,

$$\nabla^2 \vec{E} = \left(\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

A wave equation for electric wave propagating in free space !

Taking the curl of Maxwell's equation 4

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

this reduces to, $\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \left(\mu_0 \epsilon_0 \frac{\partial \vec{\nabla} \times \vec{E}}{\partial t} \right)$

[As per Vector identity $\vec{\nabla} \times (\vec{\nabla} \times \mathbf{A}) = \vec{\nabla}(\vec{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$]

For free space, $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$ (Maxwell's equation 3)

Applying the above, $\nabla^2 \vec{B} = \left(\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \right)$

The general form of magnetic wave in free space at speed of light,

$$\nabla^2 \vec{B} = \left(\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} \right), \text{ with } \mu_0 \epsilon_0 = \frac{1}{c^2}$$

A magnetic wave propagating in free space with speed of light!

A wave equation for electric wave propagating in free space

$$\nabla^2 \vec{E} = \left(\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

The general form of magnetic wave in free space at speed of light,

$$\nabla^2 \vec{B} = \left(\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} \right), \text{ with } \mu_0 \epsilon_0 = \frac{1}{c^2}$$

A general wave equation,

$$\nabla^2 \vec{A} = \left(\frac{1}{v^2} \frac{\partial^2 \vec{A}}{\partial t^2} \right), \text{ with velocity} = v$$

Maxwell's Conclusion:

1. *Both electric and magnetic waves propagate with speed of*

$$\text{light } c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$$

2. *Light waves (radiation) are electromagnetic waves*
3. *Light waves are transverse waves and electromagnetic waves are transverse in nature*
4. *Electric and magnetic fields are mutually perpendicular and perpendicular to the direction of propagation*

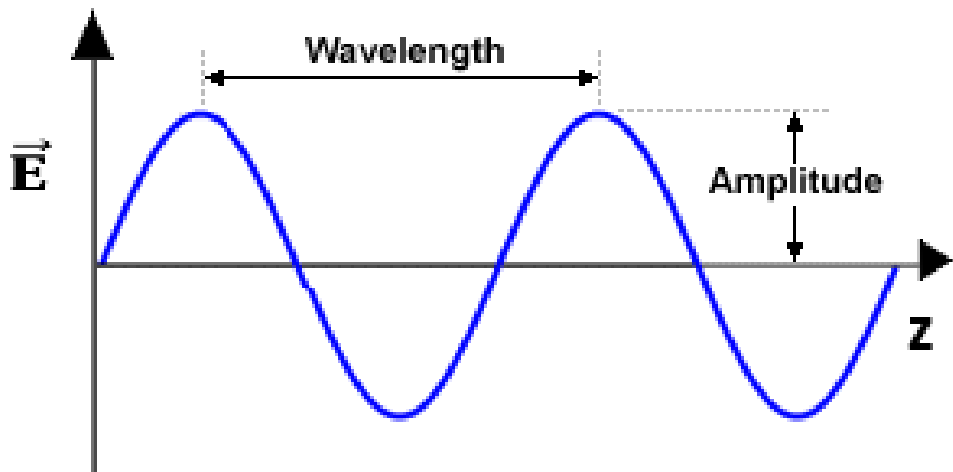
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Electromagnetic waves in free space

Analysis: E and B are mutually perpendicular to each other

Consider a 1D electric wave E_x associated with EM radiation propagating in the Z direction as,

$$E_x = E_{ox} \cos(\omega t + kz) \text{ or } E_{ox} \sin(\omega t + kz)$$



Plane E_x wave ($E_y = 0$) along z-direction

The electric field vector has only x component and other two components E_y and E_z are zero

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Electromagnetic waves in free space



The associated magnetic component of the EM wave is evaluated

as, Using Maxwell's third equation, $\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$

Evaluating curl of the electric field $\vec{\nabla} \times \vec{E} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{bmatrix}$

$$= \hat{i} \times 0 + \hat{j} * \frac{\partial E_x}{\partial z} + \hat{k} * 0 = \hat{j} \frac{\partial}{\partial z} [E_{ox} \cos(\omega t + kz)]$$

$$= -\hat{j} * k * E_{ox} \sin(\omega t + kz)$$

Thus, $-\frac{\partial \vec{B}}{\partial t} = -\hat{j} * k * E_{ox} \sin(\omega t + kz)$

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Electromagnetic waves in free space



Integrating $-\frac{\partial \vec{B}}{\partial t}$ with respect to time gives magnetic component,

$$\mathbf{B} = \hat{j} * \left(\frac{1}{\omega}\right) * E_{ox} \cos(\omega t + kz) = \hat{j} \cdot E_x * \frac{1}{c}$$

($c = \frac{\omega}{k}$, is the velocity of the radiation)

Thus,

$$E_x = \hat{i} E_{ox} \cos(\omega t + kz)$$

$$B_y = \hat{j} \left(\frac{1}{\omega}\right) * E_{ox} \cos(\omega t + kz)$$

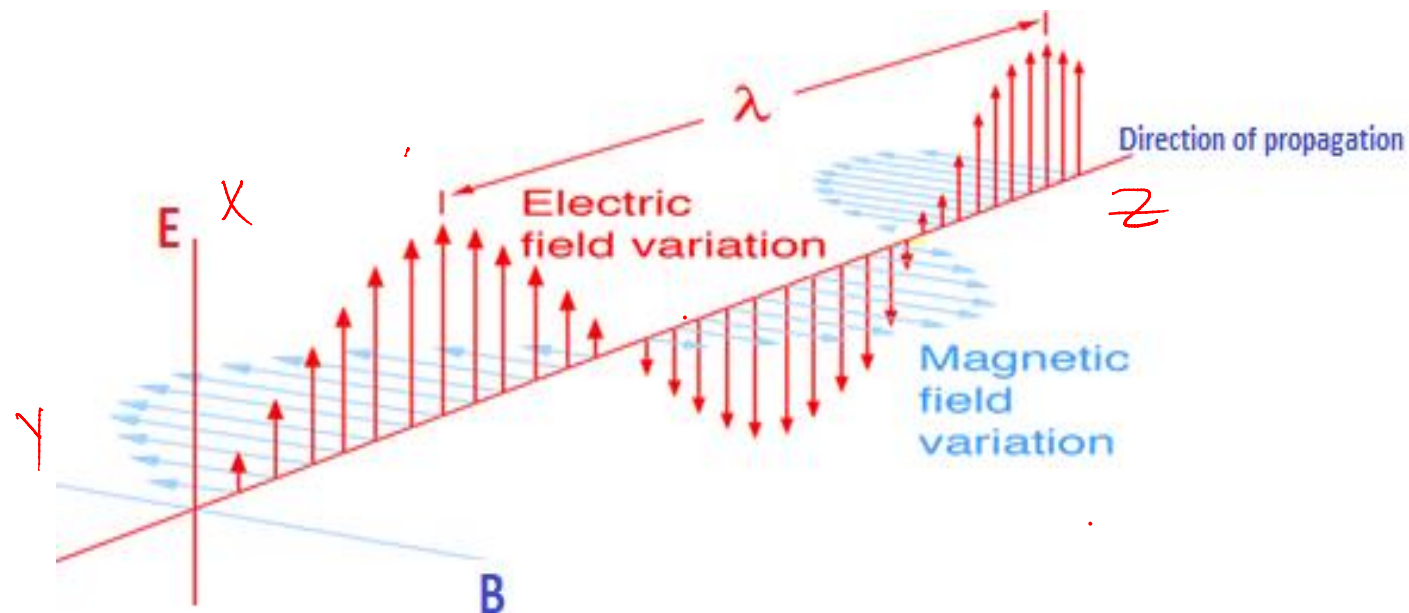
- Magnetic field is oriented in the Y-direction
- Perpendicular to the Electric field (X-direction).
- E and B are mutually perpendicular to each other.
- Magnetic field (**B**) of the EM wave is Y component
- In phase with the **E** field variations
- Phase velocity of the wave, $c = \frac{\omega}{k}$
- Magnitude of **B** wave is $\frac{1}{c}$ times the magnitude of the **E** wave

$$|B_y| = \frac{|E_x|}{c}$$

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Electromagnetic waves in free space

- EM waves have coupled **E** and **B** field components which are mutually perpendicular
- Both **E** and **B** are perpendicular to the direction of propagation



The energy density of a wave is the radiant energy per unit volume.

The electromagnetic wave consists of electric and magnetic fields which independently can store energy.

The energy content of the electric component of the wave $= \frac{1}{2} \epsilon_0 E^2$

The energy per unit volume in an electric field is dependent only on the strength of the field !

The energy content of the magnetic component of the wave

$$= \frac{1}{2} \frac{B^2}{\mu_0}$$

The energy per unit volume in a magnetic field is dependent only on the field strength!

Energy content of the electric component

$$= \frac{1}{2} \epsilon_0 E_x^2 = \frac{1}{2} \epsilon_0 E_{0x}^2 \cos^2(\omega t + kz)$$

Energy content of the magnetic component = $\frac{1}{2} \frac{B_y^2}{\mu_0}$

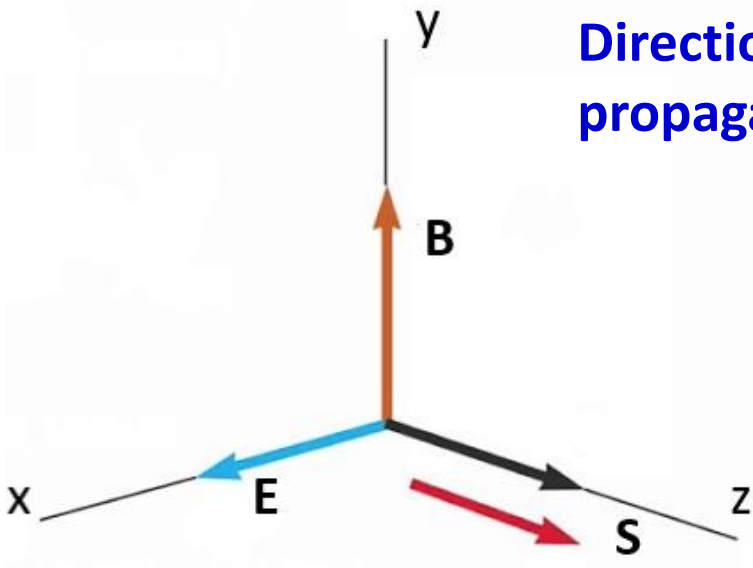
$$\begin{aligned} \text{Total energy content of the EM wave} &= \frac{1}{2} \epsilon_0 E_x^2 + \frac{1}{2} \frac{B_y^2}{\mu_0} \\ &= \frac{1}{2} \epsilon_0 E_x^2 + \frac{1}{2} \frac{E_x^2}{c^2 \mu_0} \quad \left[\text{Since, } B_y = E_x * \frac{1}{c} \right] \\ &= \epsilon_0 E_x^2, \text{ transported in the z-direction} \end{aligned}$$

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Poynting vector (Energy in an electromagnetic field)

Poynting vector (\vec{S}) describes the EM energy transported per unit time per unit volume

$$\vec{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = c^2 \epsilon_0 \mathbf{E} \times \mathbf{B}$$



Direction of Poynting vector (\vec{S}) is the direction of propagation of EM waves

- Average energy transported by an electromagnetic wave – energy transported in one cycle

- $\langle \text{Energy} \rangle = \frac{c\epsilon_0}{T} \int_0^T E_x^2 dt$
$$= \frac{c\epsilon_0}{T} \int_0^T E_{ox}^2 \sin^2(\omega t + kz) dt$$
$$= \frac{c\epsilon_0}{T} \int_0^T E_{ox}^2 \left(\frac{1 - \cos 2(\omega t + kz)}{2} \right) dt$$
$$= \frac{1}{2} \epsilon_0 c E_{ox}^2$$
$$= \frac{1}{2} c \frac{B_{oy}^2}{\mu_0}$$
$$= \frac{1}{2} \frac{E_{ox} B_{oy}}{\mu_0}$$

Average energy of EM waves

- Average energy transported by an electromagnetic wave – energy transported in one cycle

- $\langle \text{Energy} \rangle = \frac{1}{2} \epsilon_0 c E_{ox}^2 = \frac{1}{2} c \frac{B_{oy}^2}{\mu_0} = \frac{1}{2} \frac{E_{ox} B_{oy}}{\mu_0}$

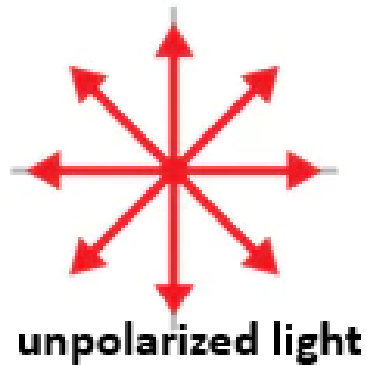
- Time averaged value of **S** gives the intensity of the wave

$$I \propto E^2$$

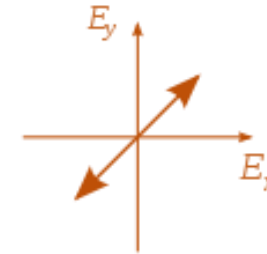
- Energy transported is dependent on the amplitude of the electric and magnetic waves
- Energy is independent of the wavelength or frequency of the waves!!

Polarisation of electromagnetic waves

- Natural light is generally unpolarized, all planes of propagation being equally probable
- Polarization of the electric wave

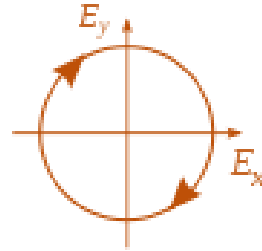


Plane polarized EM wave -
two waves in phase



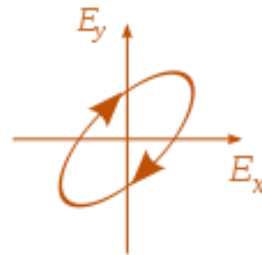
Circularly polarized EM wave –

**two waves of equal amplitudes and
out of phase by 90°**



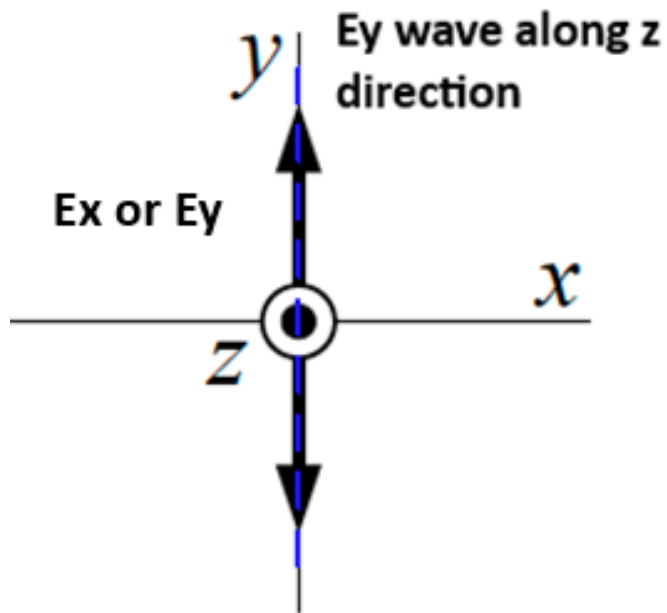
Elliptically polarized EM wave –

**two waves of unequal amplitudes and
out of phase $\neq 90^\circ$**

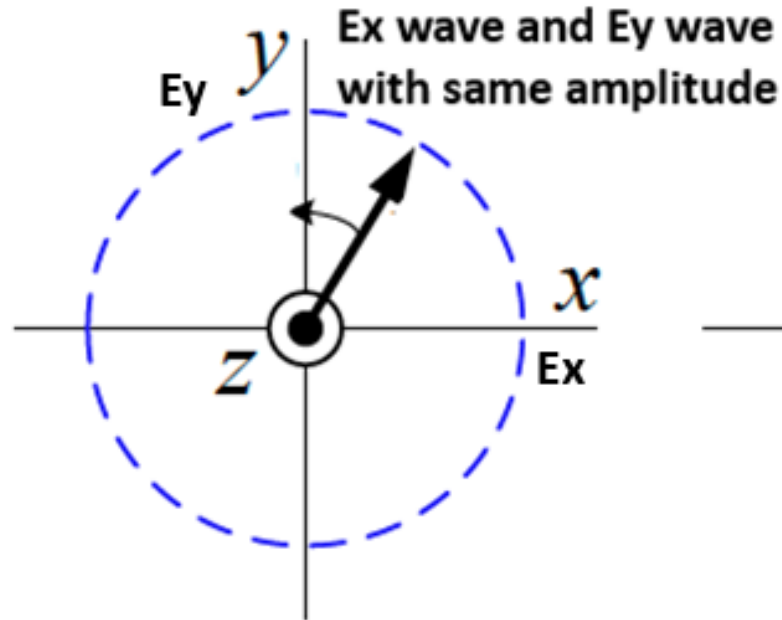


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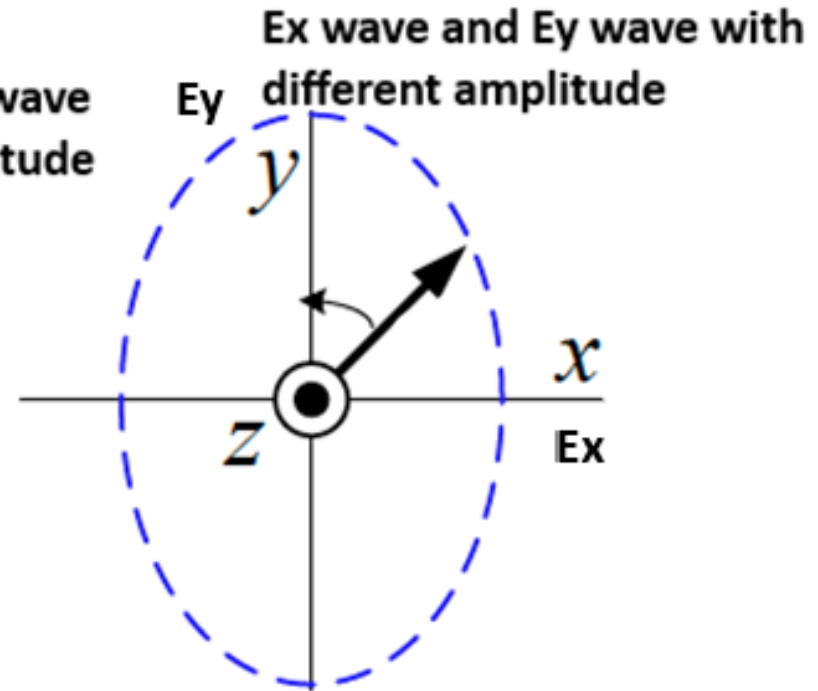
Polarization of electromagnetic waves



Linearly polarized



Circularly polarized



Elliptically polarized

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Unit I : Review of concepts leading to Quantum Mechanics



- **Overview of failure of EM wave theory**
- **Black body radiation**
- **Cavity Oscillators**
- **Classical estimation of energy density**
- **Max Planck's estimation of energy density**

- **EM Radiation** (e.g. Radio waves, microwaves, infrared, visible light, ultraviolet, x-rays and gamma radiation)-Described as mutually perpendicular sinusoidal electric and magnetic fields
- **Wave theory successfully explains the phenomena of reflection, refraction, interference, diffraction and polarization of light**

Classical wave theory could not explain many observed phenomena

- 1. Black-Body Radiation Spectrum**
- 2. Photo-electric Effect**
- 3. Spectrum of Hydrogen Emissions (Atomic Spectra)**
- 4. Compton Scattering**

Resulted in the birth and rise of Quantum Mechanics!

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Black-body radiation



Classically the interaction of radiation with matter (by absorption and emission) gives the color of the material

Gustav Robert Kirchhoff found materials which absorb all incident rays

Such a material on heating would emit all wavelengths of radiation absorbed.

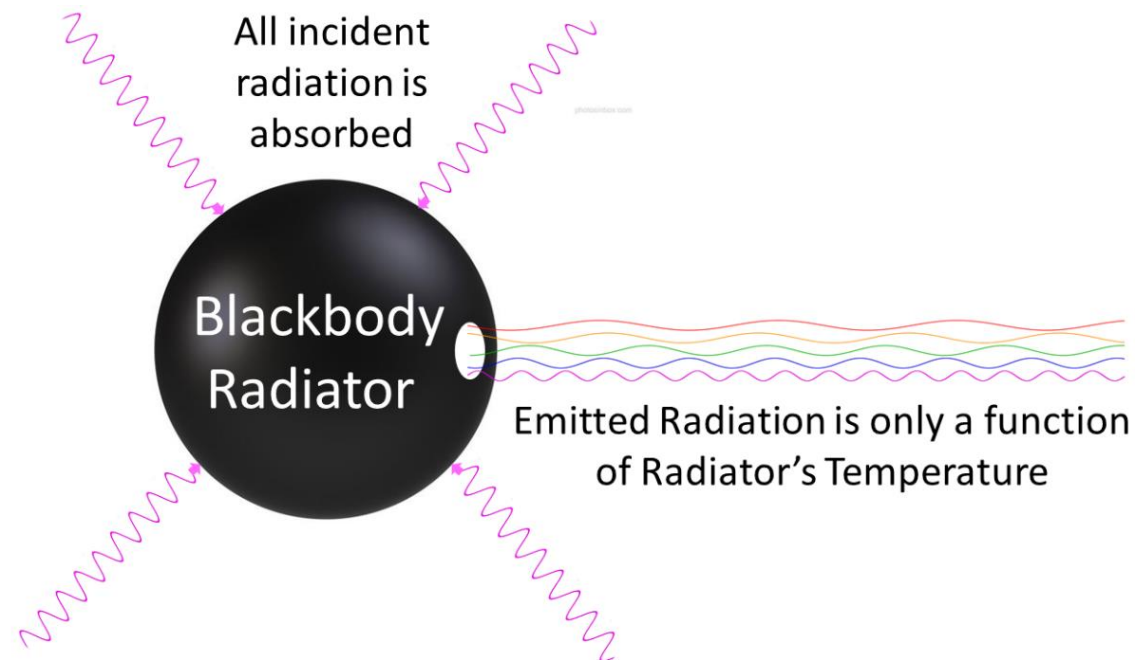
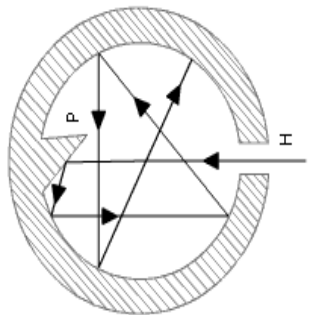
Black-body (not necessarily black!)

- **absorbs all radiations falling on it**
- **Emits all wavelengths (frequencies) as it absorbed**
- **Emissivity is unity**

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Black-body model (Cavity oscillators)

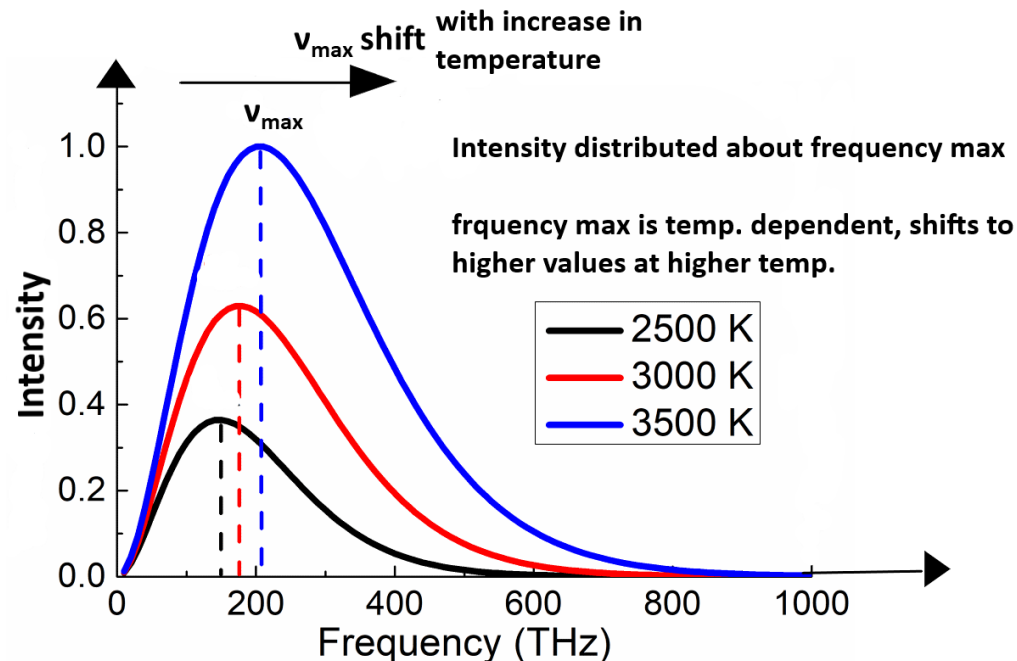
- Practically modeled as a cavity which does not allow any incident radiation to escape due to multiple reflections inside the cavity
- This cavity when heated, emit radiation of every possible frequency (rate of emission increases with temperature)



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Black-body radiation spectrum

- Radiation depends only on the temperature of the object, and not on what it is made of (**material independent**)
- As the temperature increases, emits more energy (**Area under curve**)
- As the temperature of object increases, the peak wavelength becomes shorter (**higher frequency**)
(**Blue stars are hotter than red stars!**)

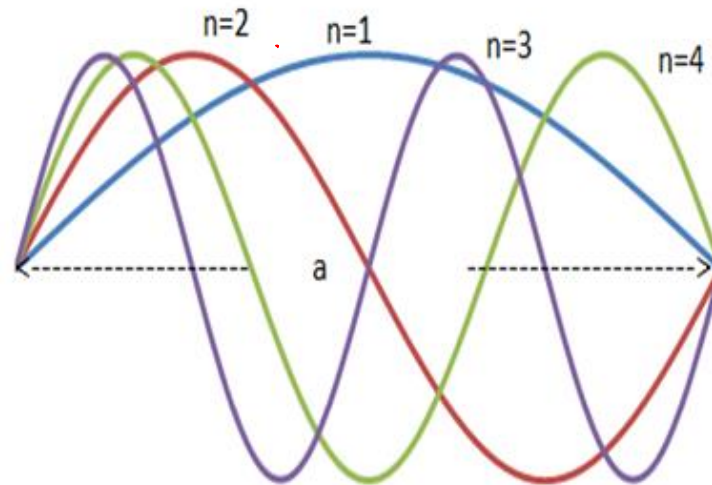


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Classical estimation of energy density

Analysis by Rayleigh-Jeans

- To understand the energy density of radiation assuming the black-body as a cavity oscillators (trapped oscillations of EM energy)
- Rayleigh and Jeans showed that the number of modes was proportional to the frequency squared
- The number of oscillators with frequencies between ν and $\nu + \delta\nu$ is calculated as $dN = \frac{8\pi}{c^3} \nu^2 d\nu$



Blackbody radiation as cavity oscillations of EM energy

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Classical estimation of energy density



The average energy of the oscillators is evaluated using

Maxwell-Boltzmann distribution law as $\langle E \rangle = k_B T$

Thus, expression for the energy density (energy per unit volume)

of radiations with frequencies between ν and $\nu + \delta\nu$ as

$$\rho(\nu) d\nu = \langle E \rangle dN = \frac{8\pi}{c^3} \nu^2 d\nu k_B T$$

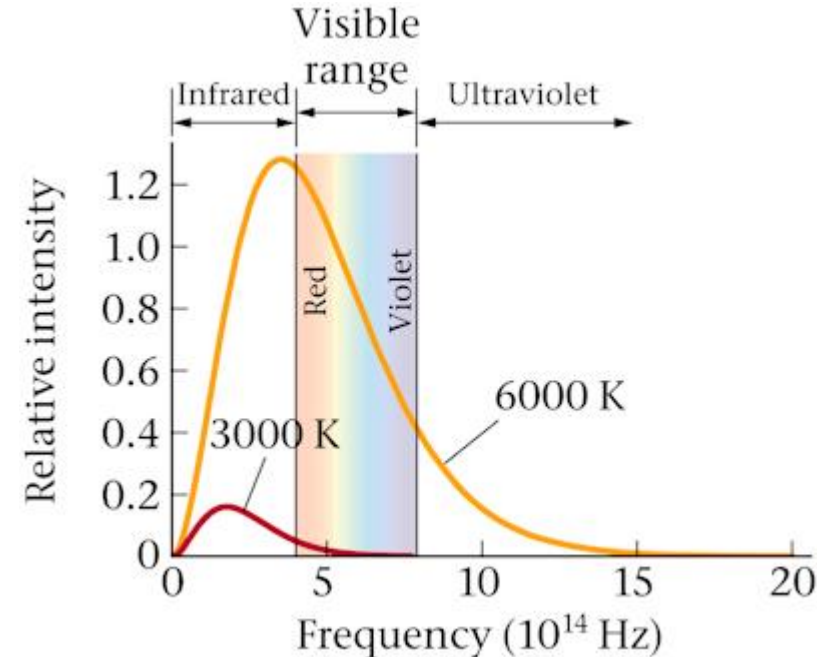
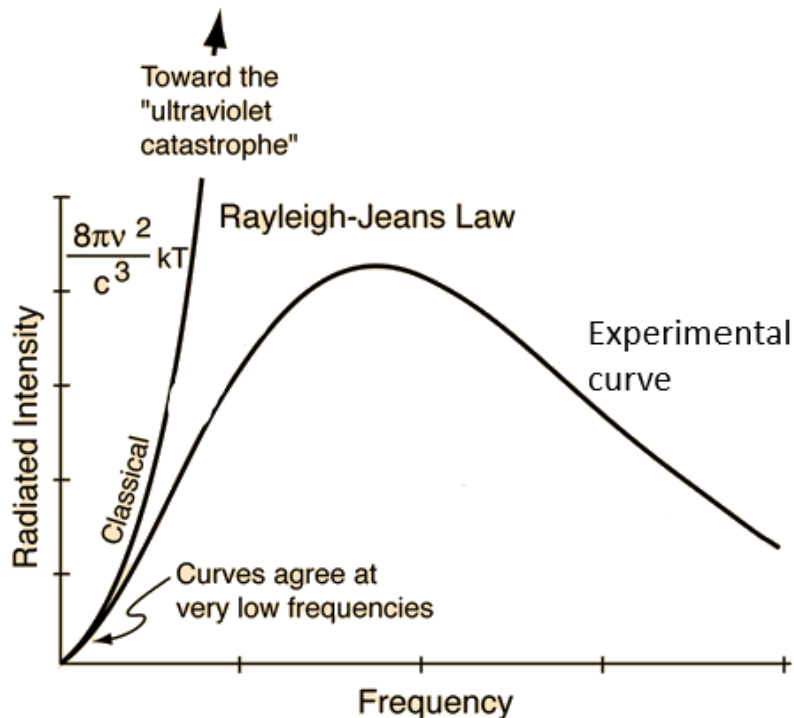
This is the Rayleigh Jeans law which is in contradiction with the experimental observations

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Failure of Rayleigh-Jeans' law

Treating EM waves as classical oscillators failed to explain the experimental observations

(intensity of radiations were found to decrease with increase in frequency - termed as **ultra-violet catastrophe**.)



Solution to the failure of classical approach to understand the energy distribution in black body by Max Planck (quantum theory of radiation, 1900)

- This theory proposed that the energy of the harmonic oscillator (oscillator model of a black body) are restricted to multiples of a fundamental natural frequency ν times a constant ($h = 6.6 \times 10^{-34} \text{ Js}$) ie., $E = nh\nu$
- Thus the radiations are from a collection of harmonic oscillators of different frequencies and the energy of the radiations has to be packets of $h\nu$

- The electrons in the black body are assumed as simple harmonic oscillators.
- An oscillator can have only discrete energies given by $E = nh\nu$
- The oscillators will not emit or absorb energy continuously
- They emit or absorb radiation in terms of quanta's of magnitude $h\nu$ discretely.

- The average energy of the oscillators were evaluated as, $\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$
- Thus, the energy density of radiations

$\rho(\nu)d\nu = \text{Number of modes} \times \text{Average energy}$

$$\langle E \rangle dN = \frac{8\pi}{c^3} \nu^2 d\nu \frac{h\nu}{e^{h\nu/kT} - 1} = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu$$

- Planck's expression gives excellent co-relation with experimental results
- *The foundation stone- for era of quantum physics!*

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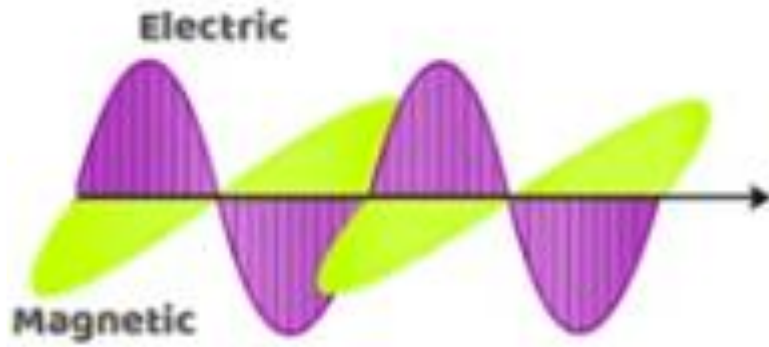
Unit I : Review of concepts leading to Quantum Mechanics

- Compton effect
- Compton shift

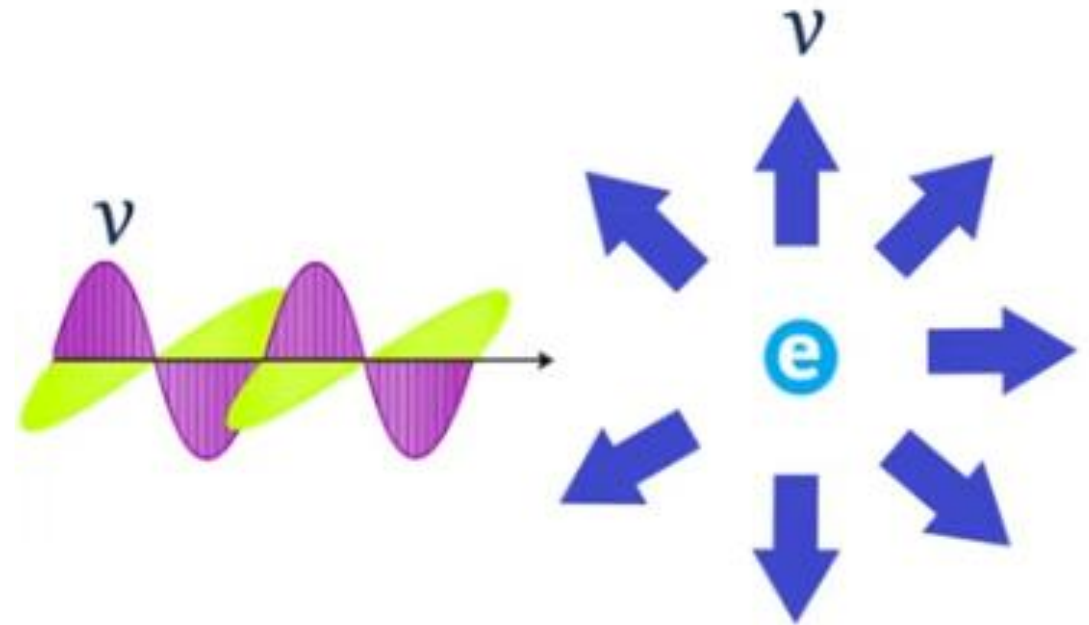


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Interaction between electromagnetic wave and matter



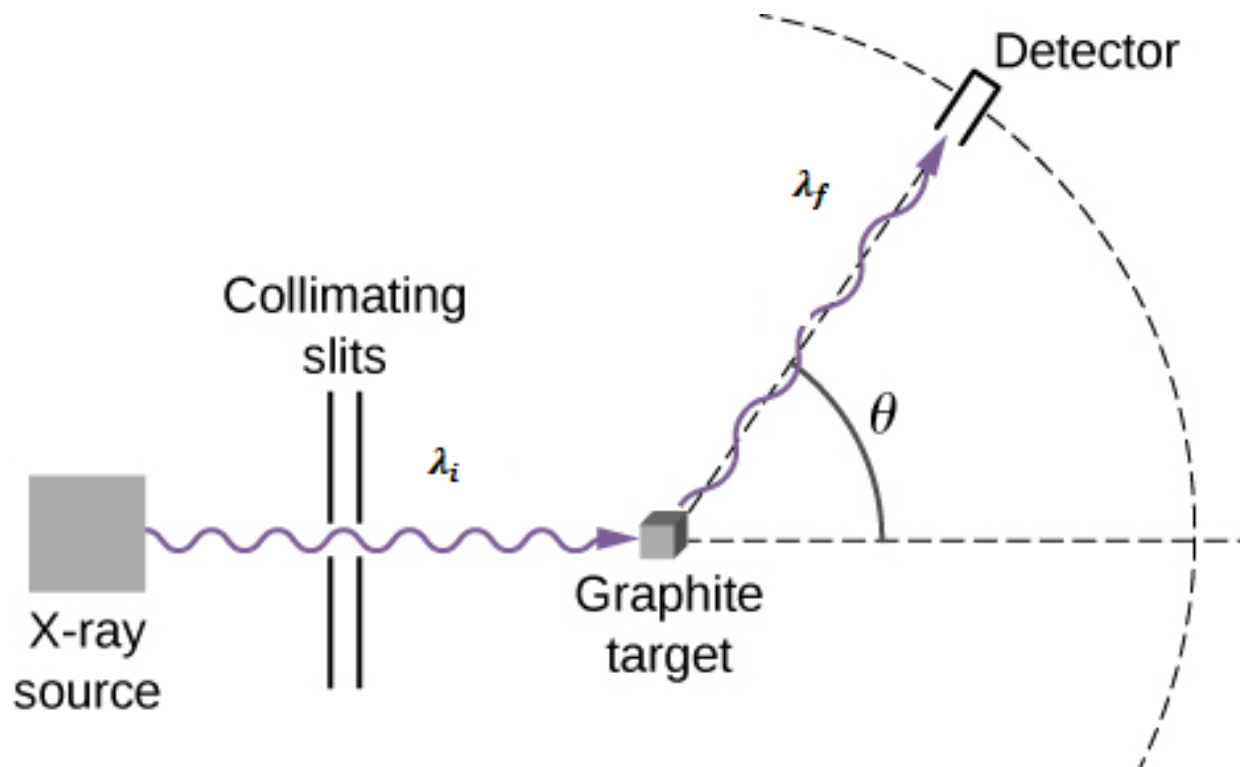
e



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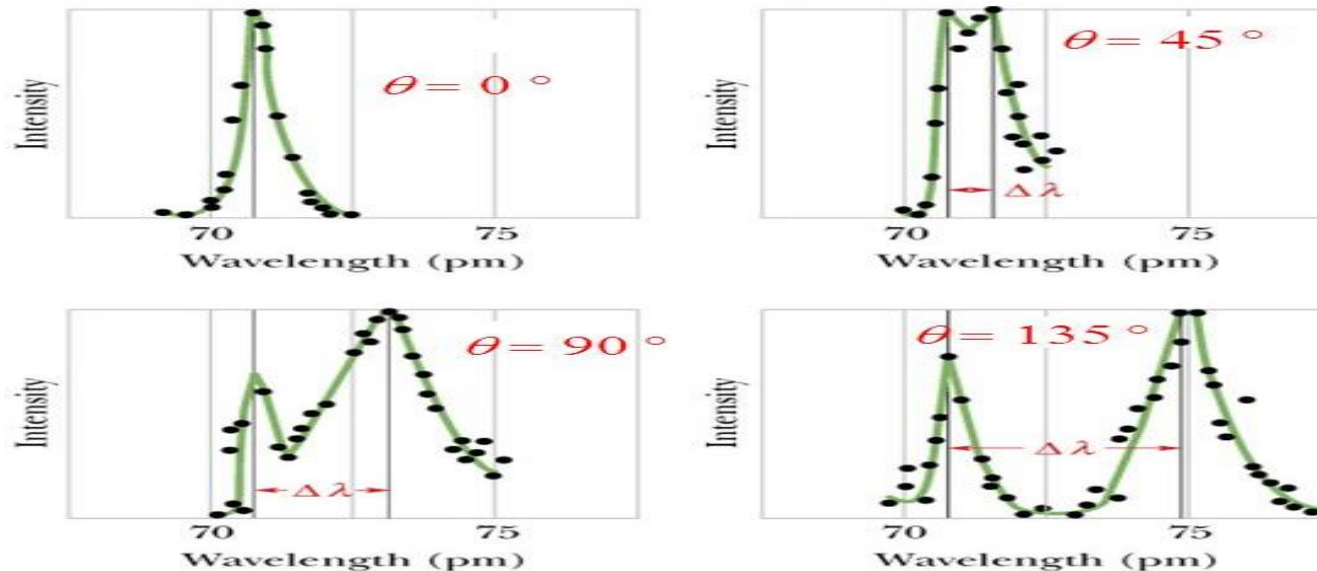
Scattering of X Rays by target materials-Compton effect

- X Ray scattering experiments with different targets
- Scattered X rays have a higher wavelength than the incident X rays



Compton shift

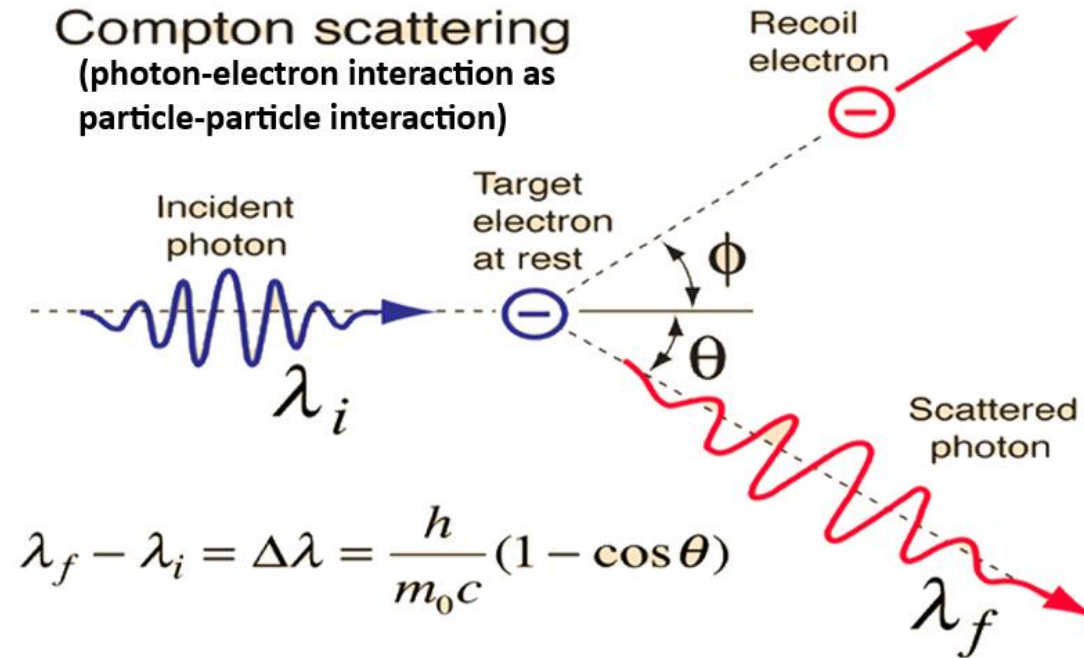
- The difference $(\lambda_f - \lambda_i)$ which indicates the enhancement in the wavelength is called the Compton shift.
- λ_i is the unmodified component
- λ_f is the modified component
- **Compton shift increases with increasing scattering angle θ .**



- Wavelength of scattered X rays is independent of the target material
- Wavelength of scattered X rays depend on the angle of scattering
- Scattering of EM waves with electrons do not explain the observed change in wavelength-*Classical explanation fails!*

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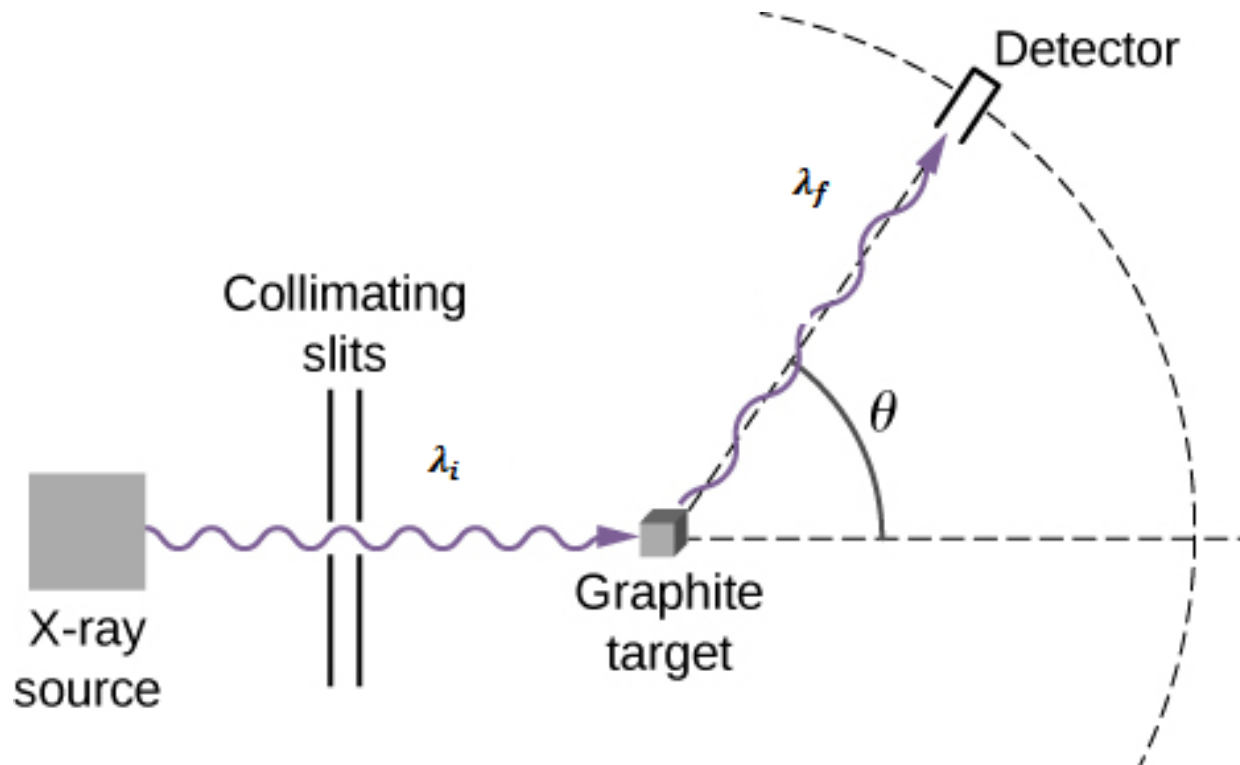
Explanation of Compton effect



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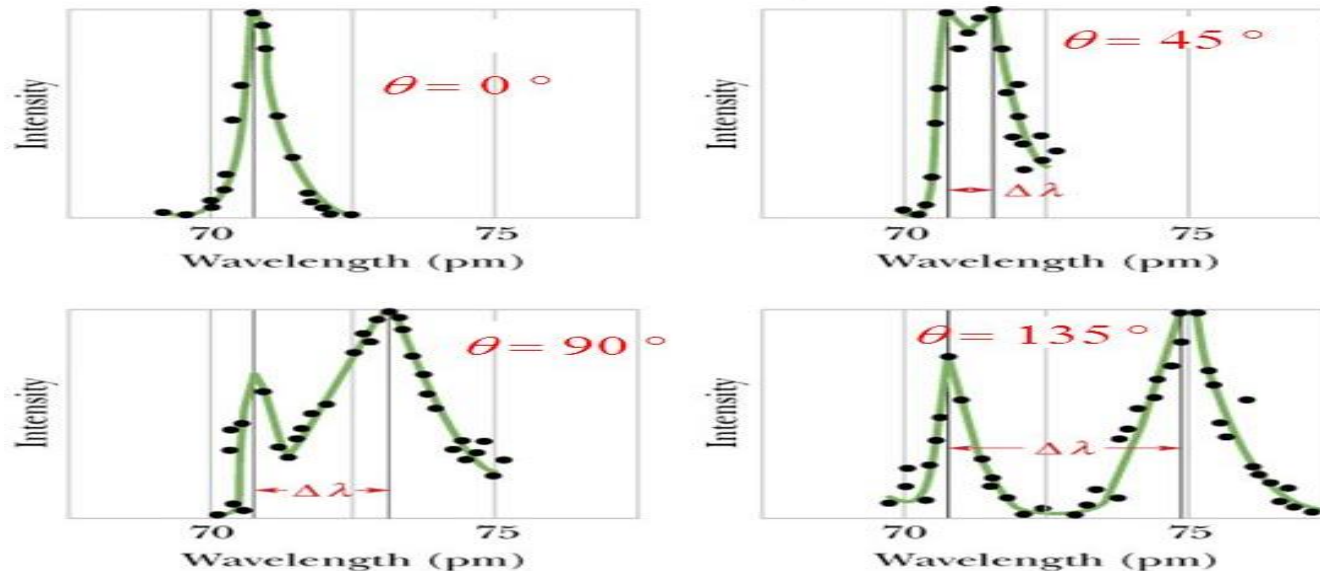
Scattering of X Rays by target materials-Compton effect

- X Ray scattering experiments with different targets
- Scattered X rays have a higher wavelength than the incident X rays



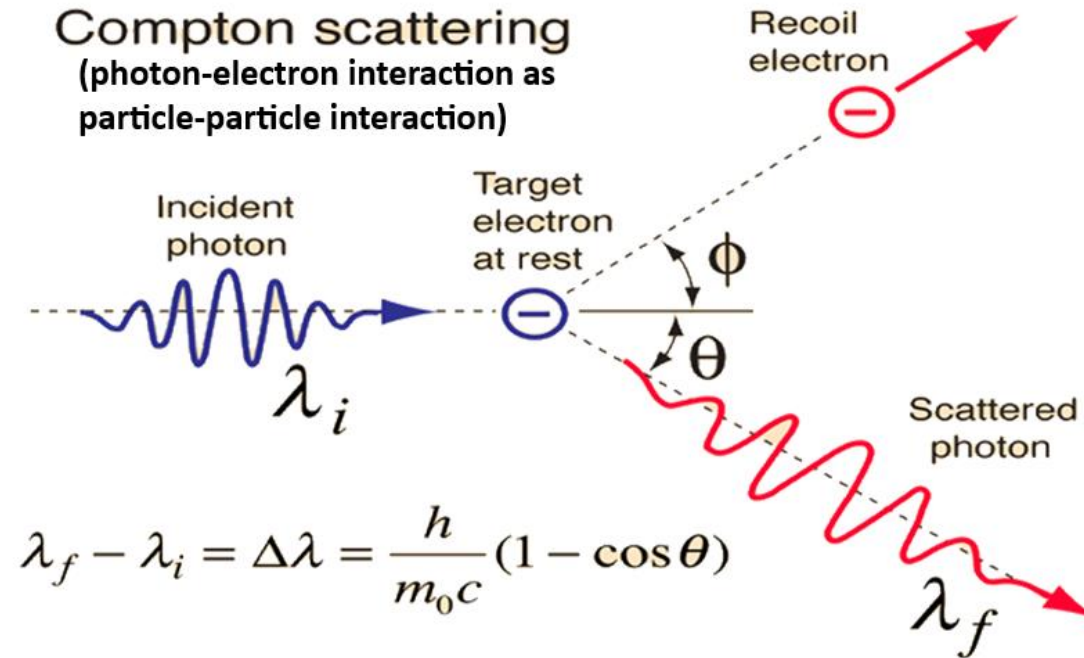
Compton shift

- The difference $(\lambda_f - \lambda_i)$ which indicates the enhancement in the wavelength is called the Compton shift.
- λ_i is the unmodified component
- λ_f is the modified component
- **Compton shift increases with increasing scattering angle θ .**



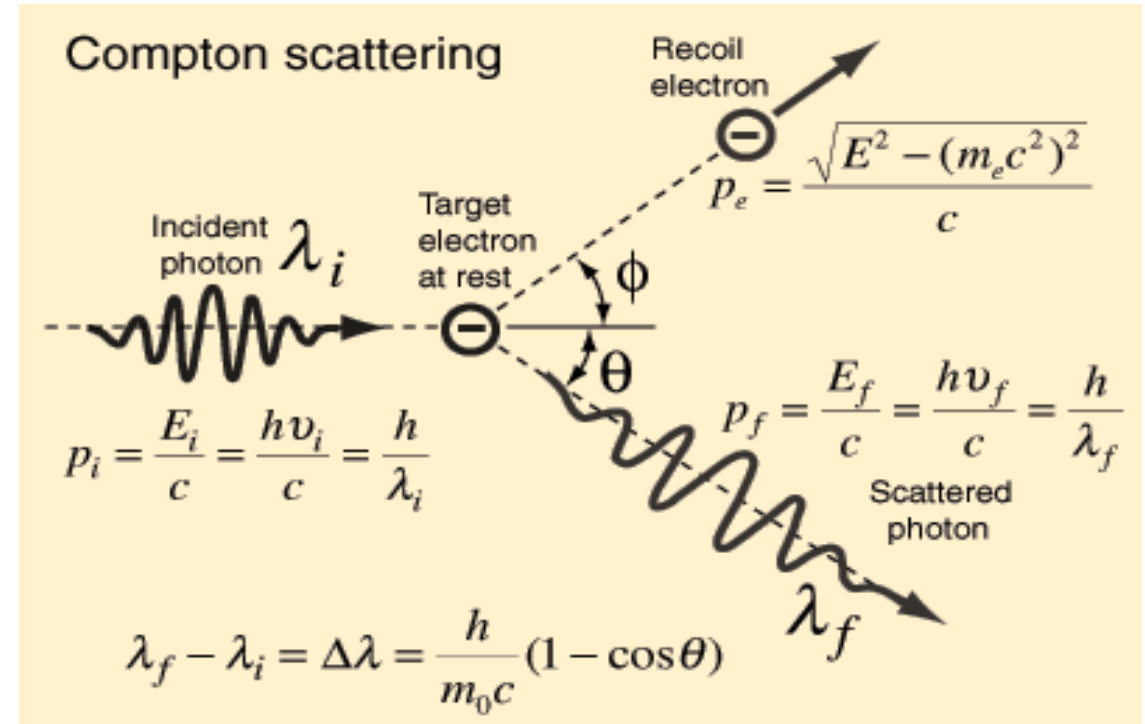
- Wavelength of scattered X rays is independent of the target material
- Wavelength of scattered X rays depend on the angle of scattering
- Scattering of EM waves with electrons do not explain the observed change in wavelength-*Classical explanation fails!*

- *X rays – part of the EM wave spectrum*
- *Interaction of radiation with matter at sub-atomic matter requires radiation to be treated as particles - photons*
- *High energy photon – particle interaction explains the scattering phenomena*
- *Wave particle duality is a reality...*



- Rest mass energy of a particle given by $E = m_0 c^2$.
- the kinetic energy of a particle with momentum p is given by pc
- The total energy of the particle is given by

$$E = \sqrt{p^2 c^2 + m_0^2 c^4}$$



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Conservation of momentum in X ray scattering

- Momentum conservation along the incident direction -

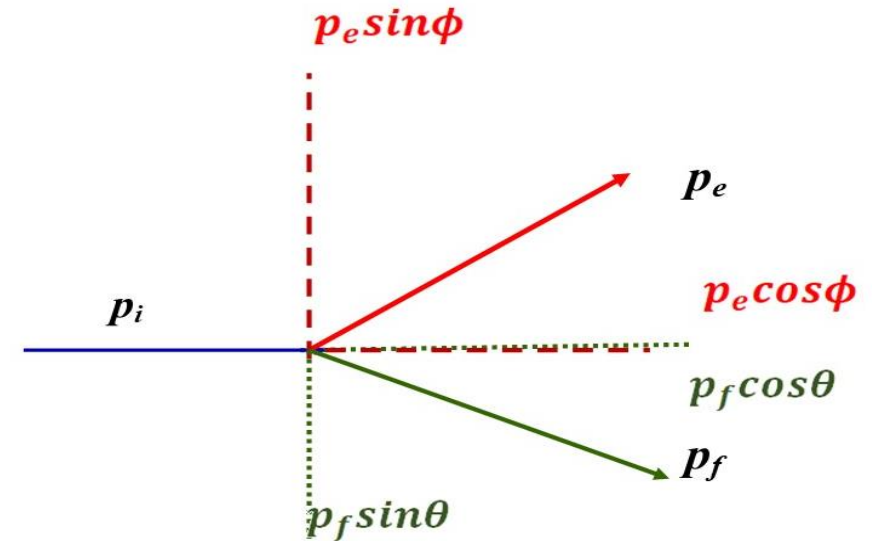
$$p_i + 0 = p_f \cos\theta + p_e \cos\phi.$$

- Momentum conservation in a perpendicular direction -

$$0 = p_f \sin\theta - p_e \sin\phi$$

(By squaring and adding the terms)

$$p_e^2 = p_i^2 + p_f^2 - 2p_i p_f \cos\theta \quad \dots (1)$$



- **Conservation of momentum before and after collision**

$$p_e^2 = p_i^2 + p_f^2 - 2p_i p_f \cos\theta \quad \dots (1)$$

- **Conservation of energy before and after collision**

$$p_i c + m_o c^2 = p_f c + \sqrt{p_e^2 c^2 + m_o^2 c^4}$$

By squaring this expression and take out the common terms
and expressing interms of

$$p_e^2 = p_i^2 + p_f^2 - 2p_i p_f + 2m_o c(p_i - p_f) \quad \dots (2)$$

- **Comparing equations 1 & 2**

- $p_e^2 = p_i^2 + p_f^2 - 2p_i p_f \cos\theta \quad \dots (1)$

$$p_e^2 = p_i^2 + p_f^2 - 2p_i p_f + 2m_0 c(p_i - p_f) \quad \dots (2)$$

- **Comparing equations 1 & 2**

$$-2p_i p_f + 2m_0 c(p_i - p_f) = -2p_i p_f \cos\theta \quad \dots (3)$$

Compton Shift

- With $p_i = \frac{h}{\lambda_i}$ and $p_f = \frac{h}{\lambda_f}$ equation 3. simplifies to

$$\lambda_f - \lambda_i = \Delta\lambda = \frac{h}{m_e c} (1 - \cos \theta)$$

- $\Delta\lambda$ is termed as the Compton shift
- $\Delta\lambda$ is independent of the incident wavelength
- $\Delta\lambda$ dependent only on the angle of scattering
- $\frac{h}{m_e c} = \lambda_c$ is termed as the Compton wave length
- For electrons $\lambda_c = 2.42 \times 10^{-12}$ m

- *Based on the analysis of dual nature of radiation-*
- *Louis de Broglie hypothesis*
 - *Moving matter should exhibit wave characteristics*
 - *Wavelength of the associated waves $\lambda = \frac{h}{p}$ where p*
is the momentum of the particle
- *Wavelengths of macro particles are extremely small to be measured*
- *Wavelengths of moving sub atomic particles can be in the measurable range ($\sim 10^{-10}m$)*

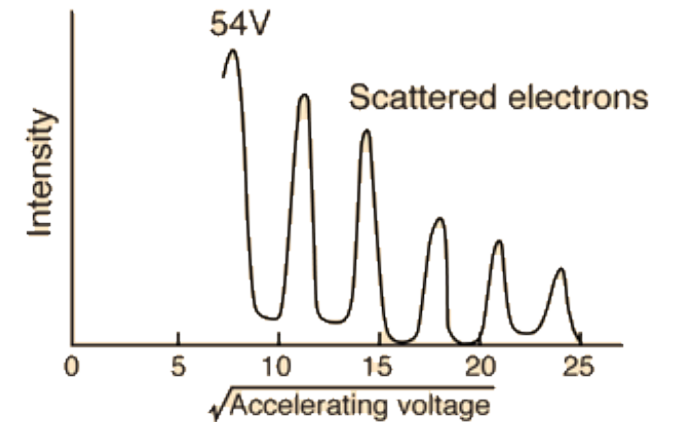
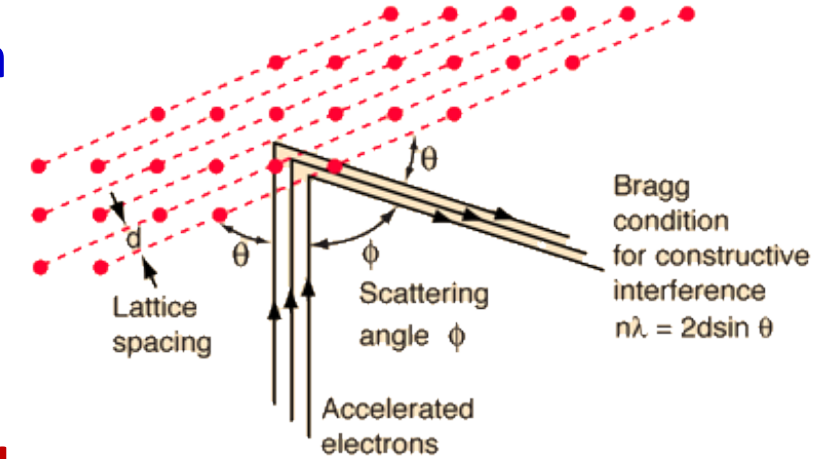
Experimental verification of de Broglie's hypothesis - Davisson and Germer's experiment (electron scattering by Ni crystals)

de Broglie wavelength $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}}$

Electron diffraction confirmed at particular settings (54 V, angle of scattering 50°)

Satisfied Bragg's law $\lambda = 2d \sin \theta$, by 'electron waves'!

Conclusion: Dual nature of matter - *matter and matter waves!*

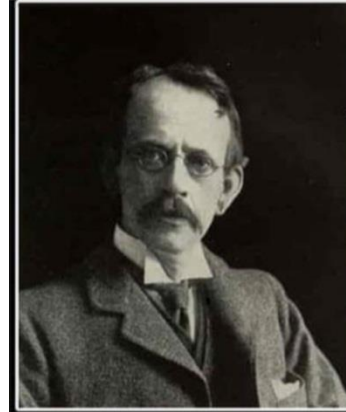


Davisson-Germer experiment

Experimental verification of de Broglie's hypothesis -
Davisson and Germer's experiment (electron
scattering by Ni crystals)

$$\text{de Broglie wavelength } \lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}}$$

Conclusion: Dual nature of matter - *matter and matter waves!*



J.J THOMSON WON THE
NOBEL PRIZE IN
PHYSICS (1906) FOR
SHOWING ELECTRONS
WERE PARTICLES.



**HIS SON GEORGE
THOMSON** WON THE
NOBEL PRIZE (1937)
WHEN HE SHOWED
ELECTRONS WERE
WAVES.

Young's double slit experiment

- *Young's classic double slit experiment on interference and diffraction of radiations*
- *Characteristic wave experiment*

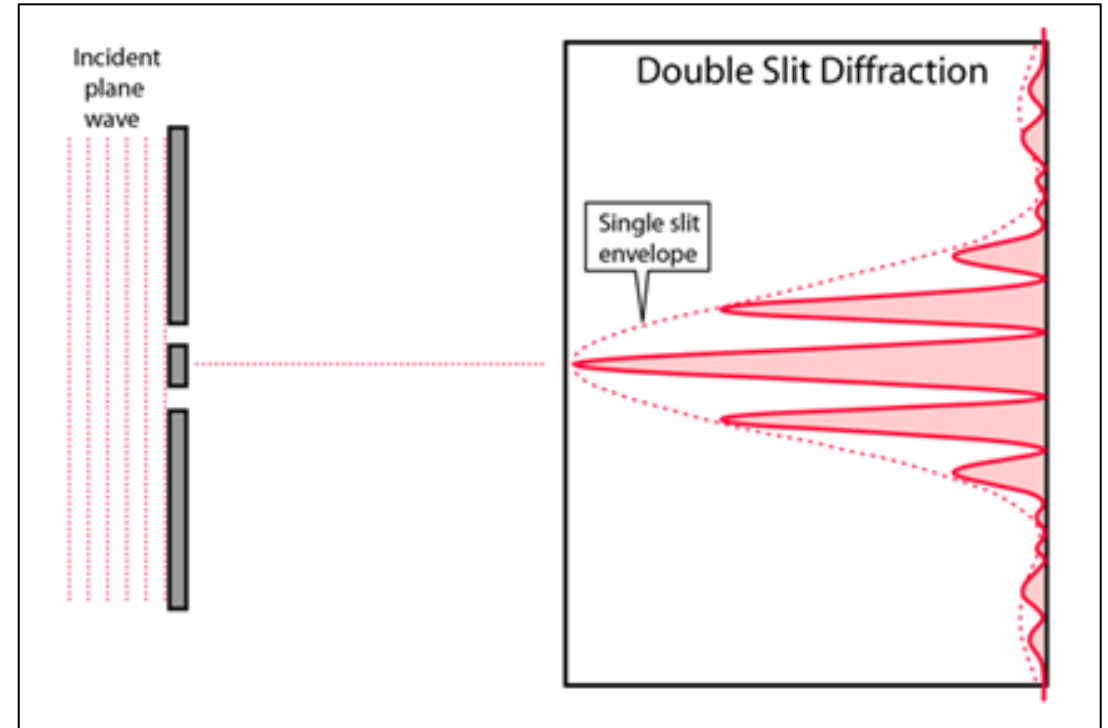
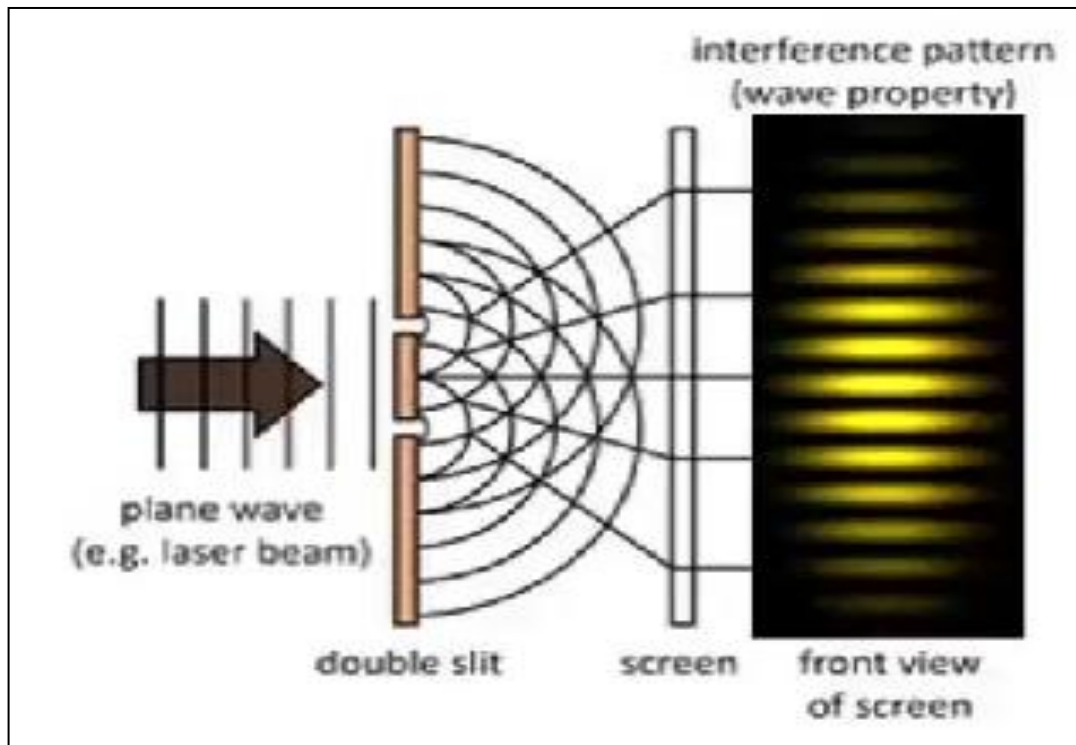
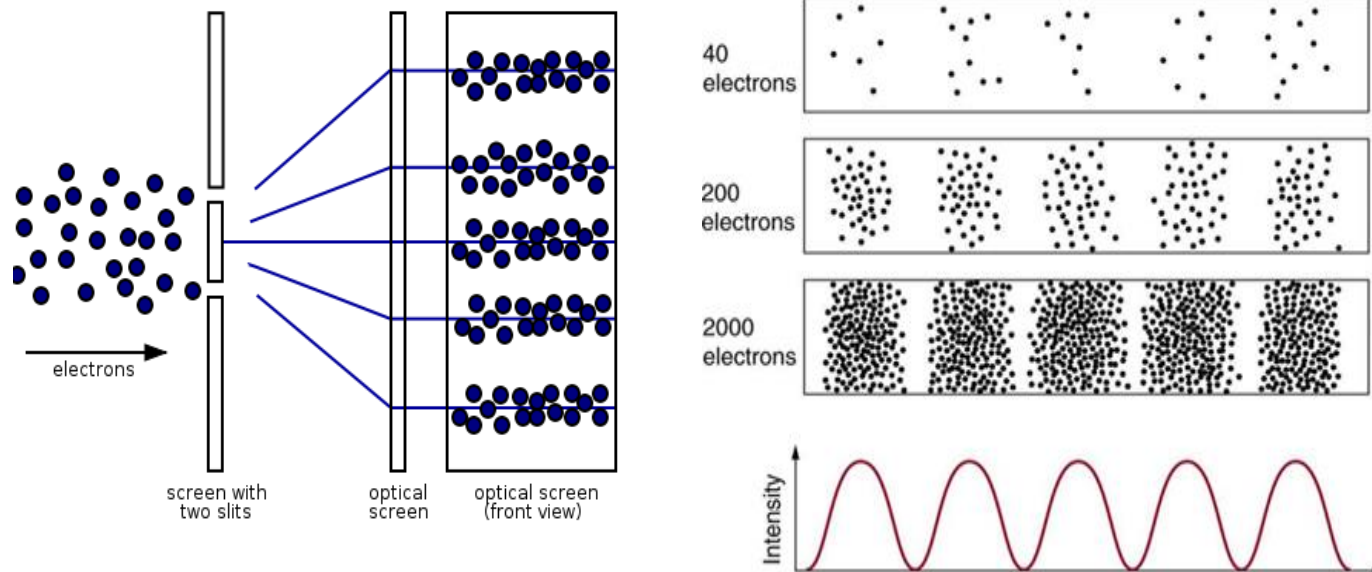


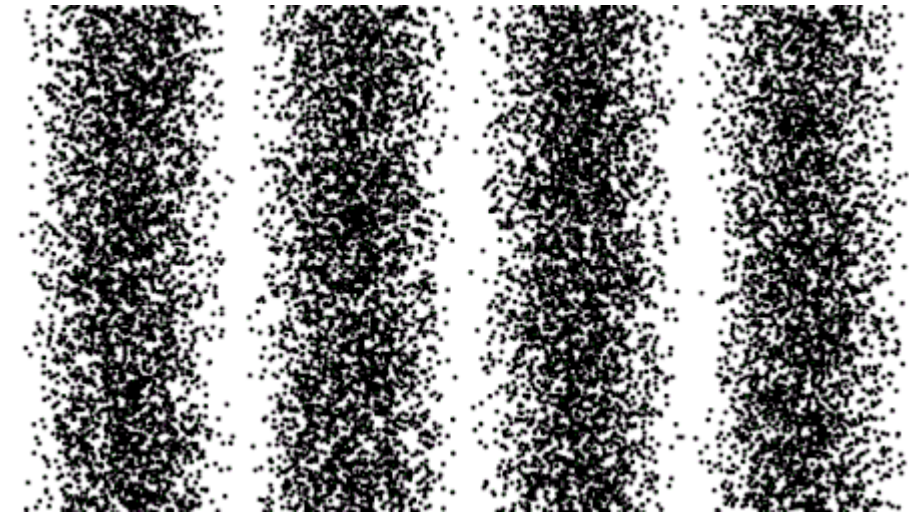
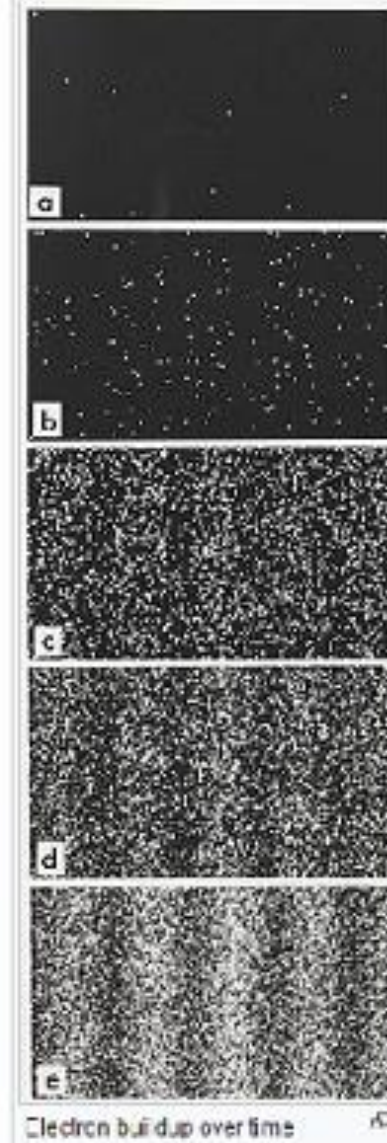
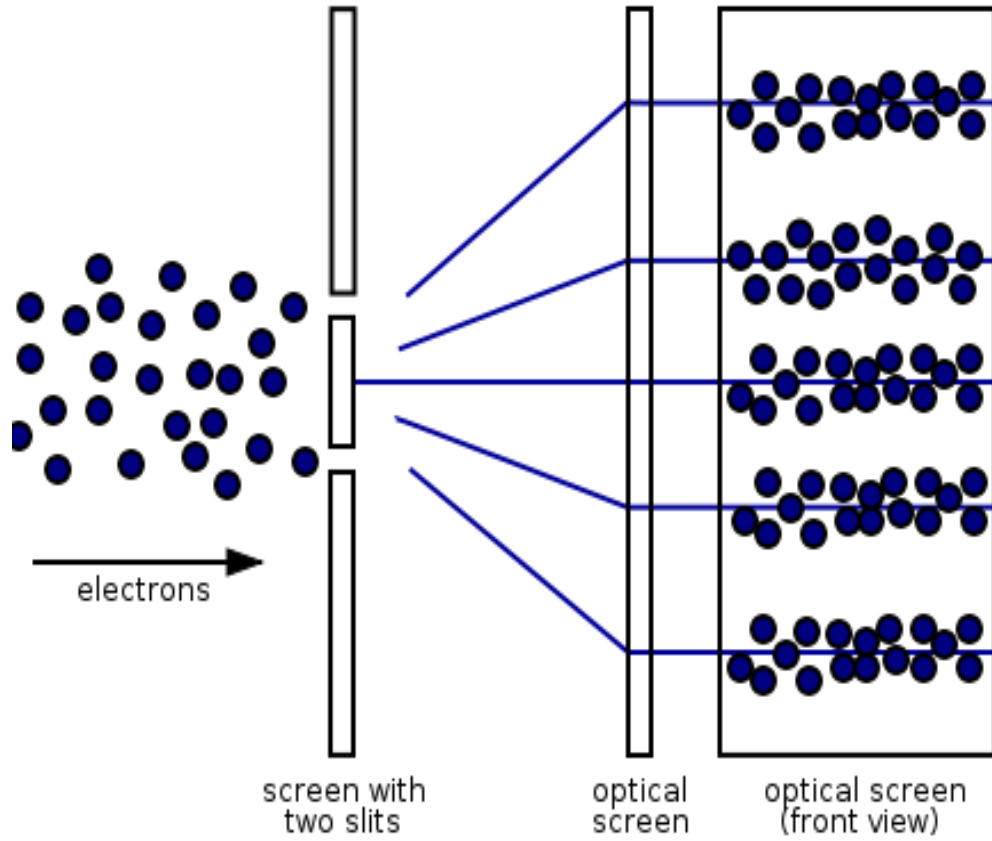
Image courtesy [Hyperphysics, Wikipedia](#)

Double slit experiment with electrons

- *Diffraction is characteristic wave phenomenon*
- *Double slit experiment with a particle (single electrons or photons – one at a time) show wave nature – Particle diffraction!*



Double slit experiment with single particles

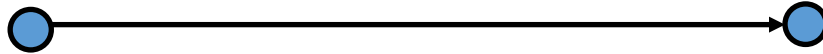


Concept of Matter waves

- *Position and momentum are the two generalized parameters needed to describe the state of any system*
- *Position and momentum are conjugate parameters*

(x_1, t_1)

(x_2, t_2)

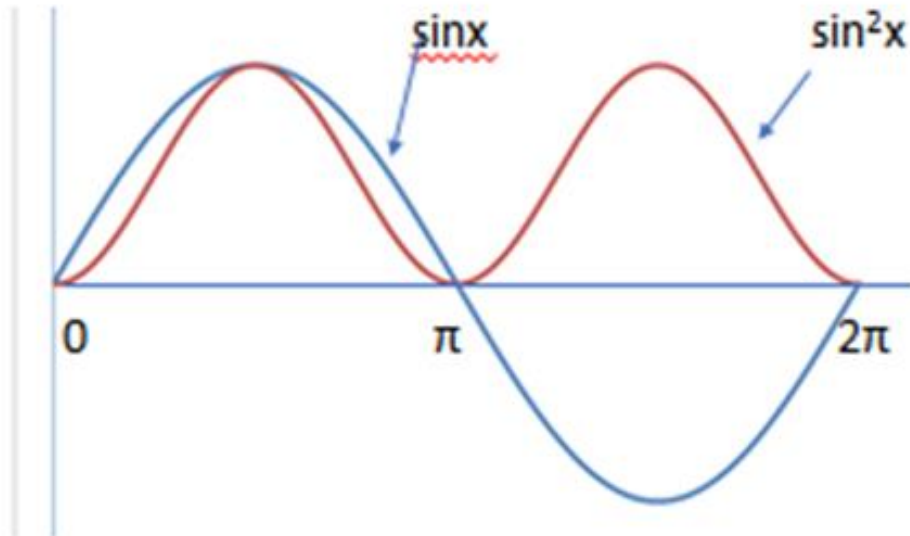


$$v = \frac{dx}{dt}$$

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Concept of matter waves

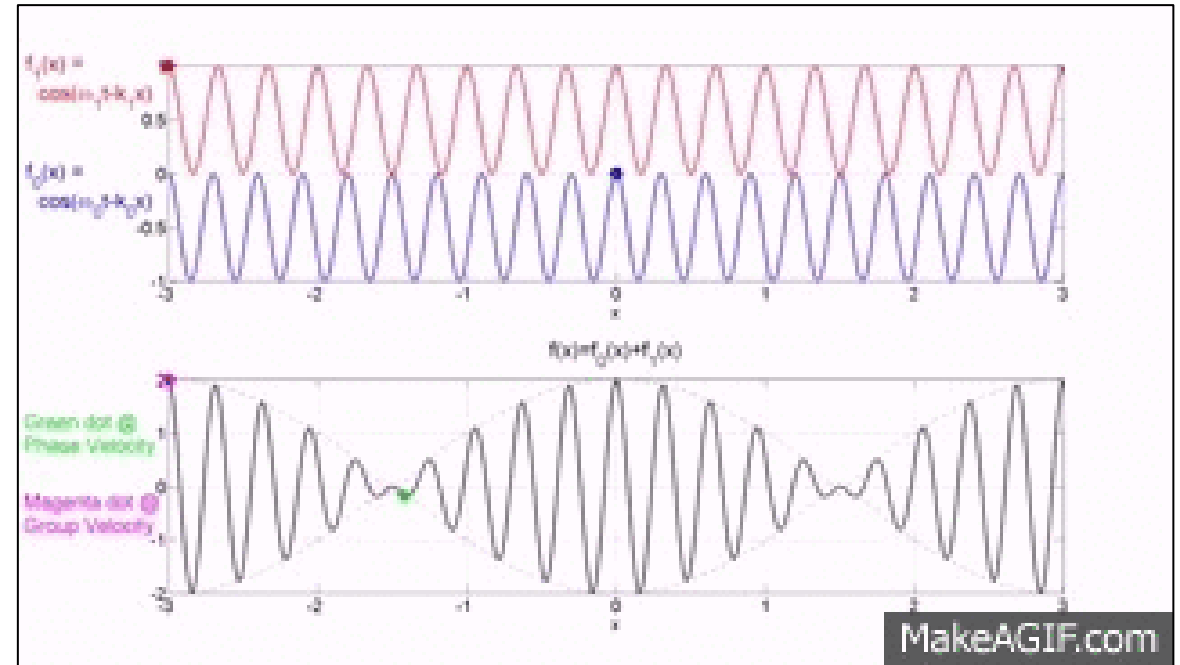
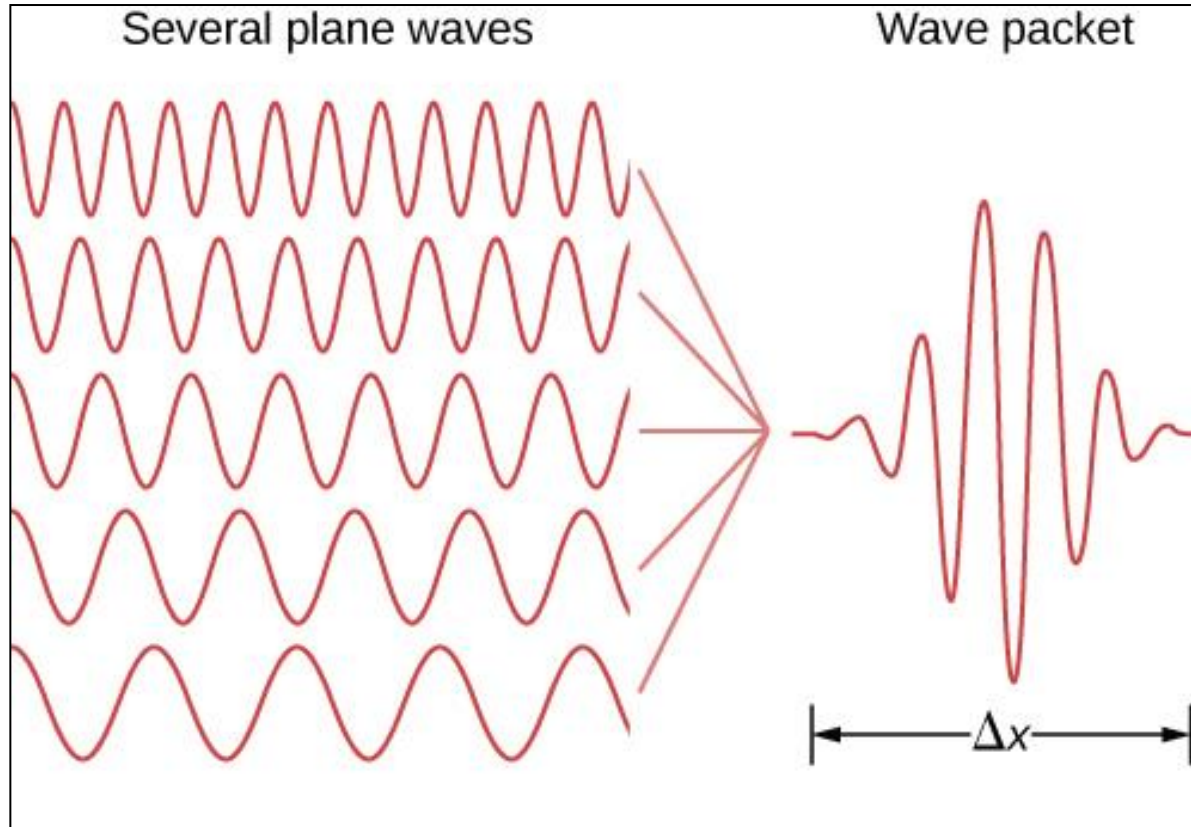
- Need a mathematical concept to describe matter waves
- Any representative wave should be able to give information about the position and momentum of the system
- Simple sine or cosine waves fall short (Momentum can be inferred from wavelengths $p = h/\lambda$ but Position is not well defined)



Important - the sine wave is not the path of the particle

Need for a new wave format for matter waves !

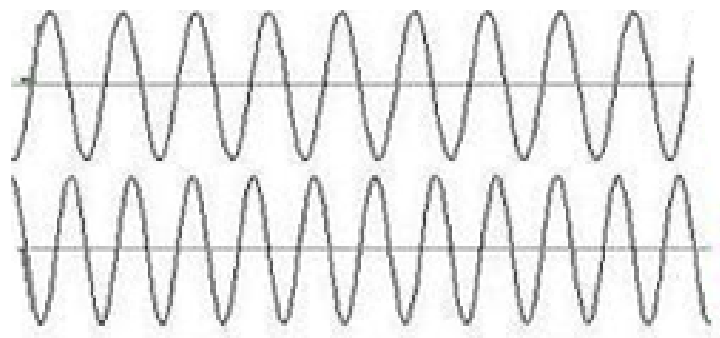
Wave Packet



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superposition of waves

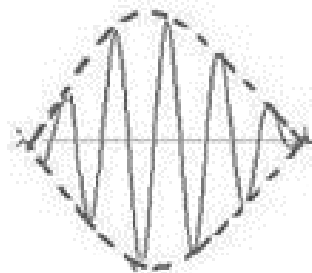
- Superposition of two waves - wave packets describe matter waves
- *Wave packets* describe matter waves - with a defined wavelength and an amplitude maximum
- Provide information about both **position** and **momentum**



Wave 1, y_1

+

Wave 2, y_2

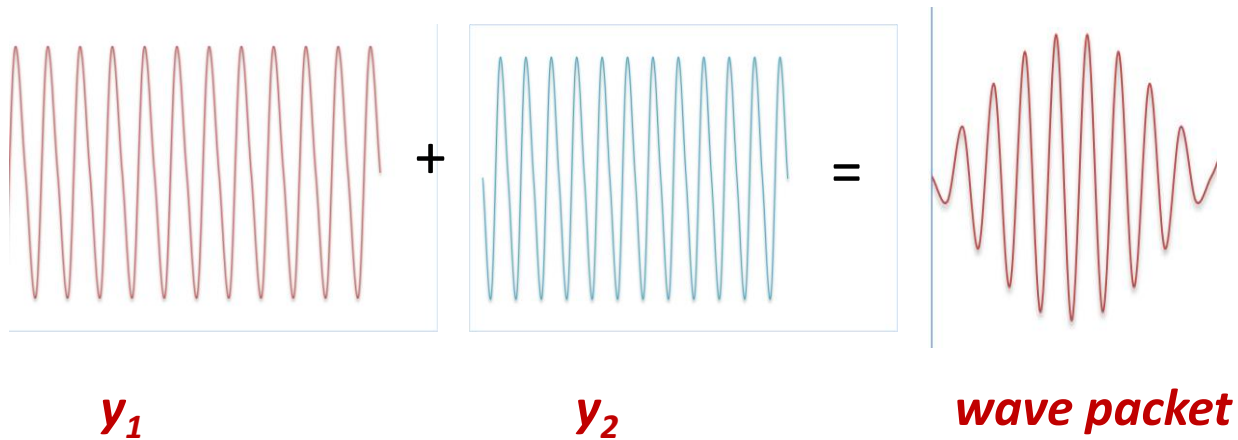


Wave packet, $y = y_1 + y_2$

superposition of two waves to a wave packet

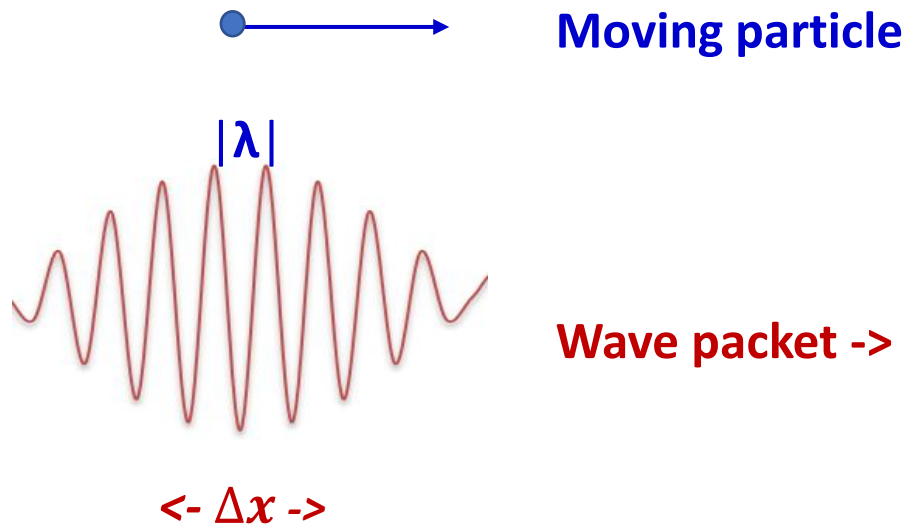
Wave Packets

- $y_1 = A \sin(\omega t - kx)$
- $y_2 = A \sin\{(\omega + \Delta\omega) t - (k + \Delta k)x\}$
- **Superposition**
- $y = y_1 + y_2 = 2A \sin(\omega t - kx) \cdot \cos\left(\frac{\Delta\omega t - \Delta kx}{2}\right)$



Wave Packets

- From k we can infer λ - which defines momentum
- The spread around the central maximum can be the approximate position of the particle

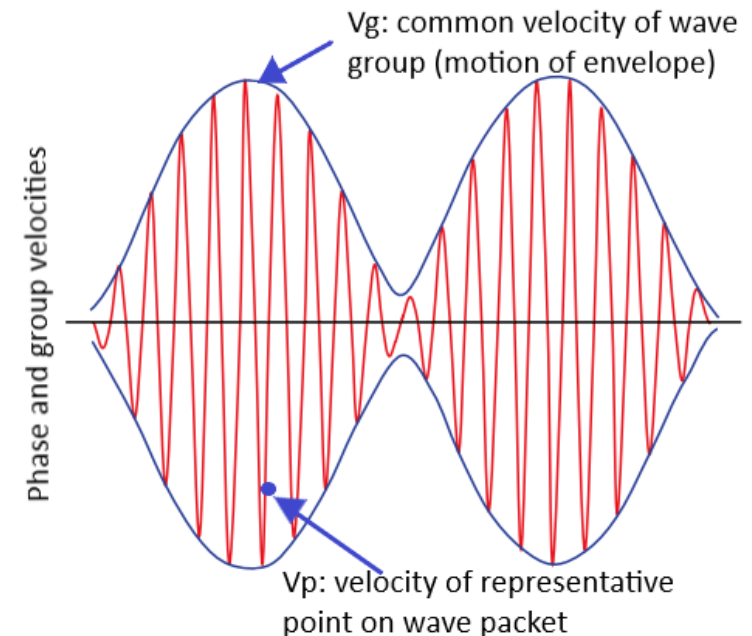


- $y = y_1 + y_2 = 2A \sin(\omega t - kx) \cdot \cos\left(\frac{\Delta\omega t - \Delta kx}{2}\right)$
- *The phase velocity of the wave packet is the velocity of a representative point on the wave packet*

$$v_{ph} = \frac{\omega}{k}$$

- *The group velocity of the wave packet is the velocity of common velocity of the superposed wave group*

$$v_g = \frac{d\omega}{dk}$$



- *The energy of the wave packet*

$$E = h\nu = \frac{h\omega}{2\pi} = \hbar\omega$$

- *The momentum of the particle*

$$p = \frac{h}{\lambda} = \frac{h \cdot 2\pi}{2\pi \cdot \lambda} = \hbar k$$

- *Group velocity*

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{d}{dp} \left(\frac{p^2}{2m} \right) = \frac{p}{m} = v_{particle}$$

- *Group velocity is reflecting the particle velocity*

- **Group velocity**

$$\begin{aligned}v_g &= \frac{d}{dk}(\omega) = \frac{d}{dk}(v_{ph}k) \\ &= v_{ph} + k \frac{dv_{ph}}{dk} = v_{ph} + k \frac{dv_{ph}}{d\lambda} \frac{d\lambda}{dk} \\ &= v_{ph} - \lambda \frac{dv_{ph}}{d\lambda}\end{aligned}$$

- **Group velocity is dependent on the phase velocity and how the phase velocity changes with wavelength**

Group and Phase velocity relation

Note: For a dispersive medium (Non-isotropic medium)

$$V_g \neq V_p$$

$$V_g = V_p - \lambda \frac{dV_p}{d\lambda}$$

② For a non-dispersive medium (Isotropic medium)

$$V_g = V_p$$

i.e. $-\lambda \frac{dV_p}{d\lambda} = 0$ but $\lambda \neq 0$

$\therefore \frac{dV_p}{d\lambda} = 0$ (Phase velocity (V_p) does not change with λ)

Group and Phase velocity relation

Group velocity = Phase velocity

$$v_g = v_{ph}$$

$$-\lambda \frac{dv_{ph}}{d\lambda} = 0$$

- *Phase velocity does not change with wavelength*
- *The medium is non dispersive*
- *A dispersive medium is one in which*

Group velocity <> Phase velocity

➤ $v_g < v_{ph}$

➤ $v_g > v_{ph}$

- $v_g < v_{ph}$ - group velocity is half the phase velocity
- $v_g = v_{ph}/2$
- $v_g = v_{ph} - \lambda \frac{dv_{ph}}{d\lambda}$
- $\frac{dv_p}{v_{ph}} = \frac{1}{2} \frac{d\lambda}{\lambda}$ on integration yields
- $\ln(v_{ph}) \propto \ln \sqrt{\lambda}$ or $v_{ph} \propto \sqrt{\lambda}$
- This implies that the phase velocity is proportional to the square root of the wavelength

$$v_g = \frac{v_{ph}}{2}$$

Substituting in $v_g = v_{ph} - \lambda \frac{dv_{ph}}{d\lambda}$

$$\frac{v_{ph}}{2} = v_{ph} - \lambda \frac{dv_{ph}}{d\lambda}$$

$$\lambda \frac{dv_{ph}}{d\lambda} = v_{ph} - \frac{v_{ph}}{2} = \frac{1}{2} v_{ph}$$

$$\frac{dv_{ph}}{v_{ph}} = \frac{1}{2} \frac{d\lambda}{\lambda}$$

Integrating, $\ln(v_{ph}) = \frac{1}{2} \ln(\lambda)$

$$\ln(v_{ph}) = \ln \lambda^{1/2}$$

$$e^{\ln(v_{ph})} = e^{\ln \sqrt{\lambda}}$$

$$v_{ph} \propto \sqrt{\lambda}$$

Group and Phase velocity relations

- $v_g > v_{ph}$ - **group velocity is twice the phase velocity**
- $v_g = 2v_{ph}$
- $v_g = v_{ph} - \lambda \frac{dv_{ph}}{d\lambda}$
- $\frac{dv_p}{v_{ph}} = -\frac{d\lambda}{\lambda}$ **on integration yields**
- $\ln(v_{ph}) \propto \ln \frac{1}{\lambda}$ or $v_{ph} \propto \lambda^{-1}$
- **This implies that the phase velocity is inversely proportional to the wavelength**

$$v_g = 2v_{ph}$$

Substituting in $v_g = v_{ph} - \lambda \frac{dv_{ph}}{d\lambda}$

$$2v_{ph} - v_{ph} = -\lambda \frac{dv_{ph}}{d\lambda}$$

$$v_{ph} = -\lambda \frac{dv_{ph}}{d\lambda}$$

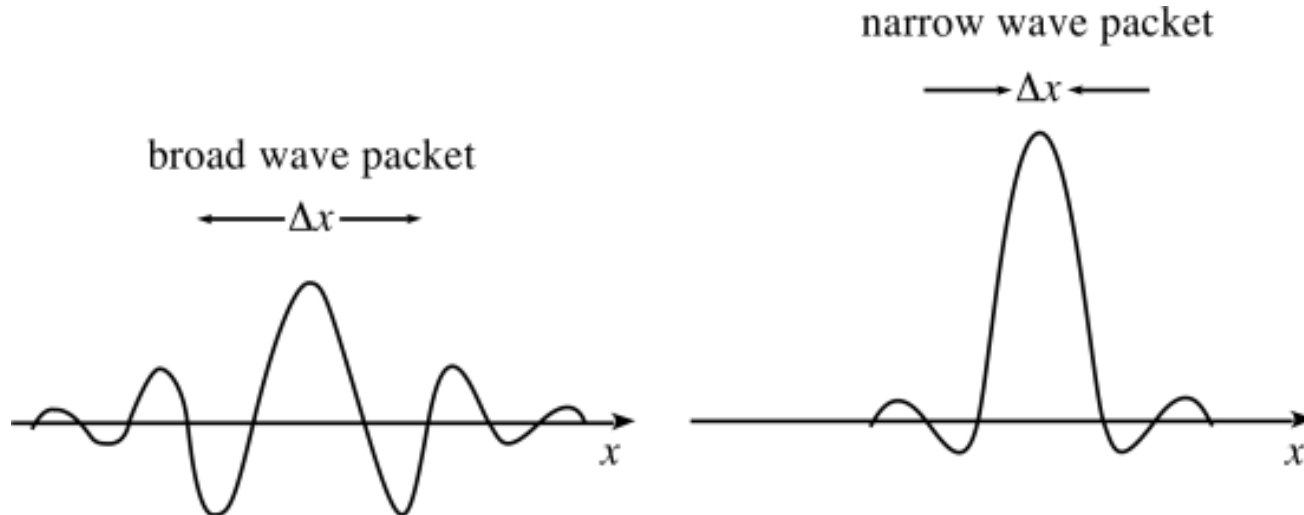
$$\frac{dv_{ph}}{v_{ph}} = -\frac{d\lambda}{\lambda}$$

Integrating, $\ln(v_{ph}) \propto \ln(1/\lambda)$

$$v_{ph} \propto \lambda^{-1}$$

Analysis of Wave Packets

- *Wave packets are formed by the superposition of waves*
 - *Wave packets have an inherent component of uncertainties*
 - *A broad wave packet has a high uncertainty in position, but a high accuracy in momentum*
 - *A narrow wave packet results in a high accuracy in position and a high uncertainty in the momentum*
-



Heisenberg's Uncertainty Principle

Position momentum uncertainty:

The position and momentum of a particle cannot be determined simultaneously with unlimited precision.

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

Δx - *the uncertainty in the position*

Δp - *the uncertainty in the momentum*

Energy time uncertainty:

The energy and life time of a particle in a state cannot be determined simultaneously with unlimited precision.

ΔE - the uncertainty in the energy of the particle

Δt - the uncertainty in the life time of the particle in the state

The product of the uncertainties $\Delta E \cdot \Delta t \geq \frac{\hbar}{2}$

Uncertainty relation for circular motion:

The angular position and angular momentum of a particle in a circular motion cannot be determined simultaneously with unlimited precision.

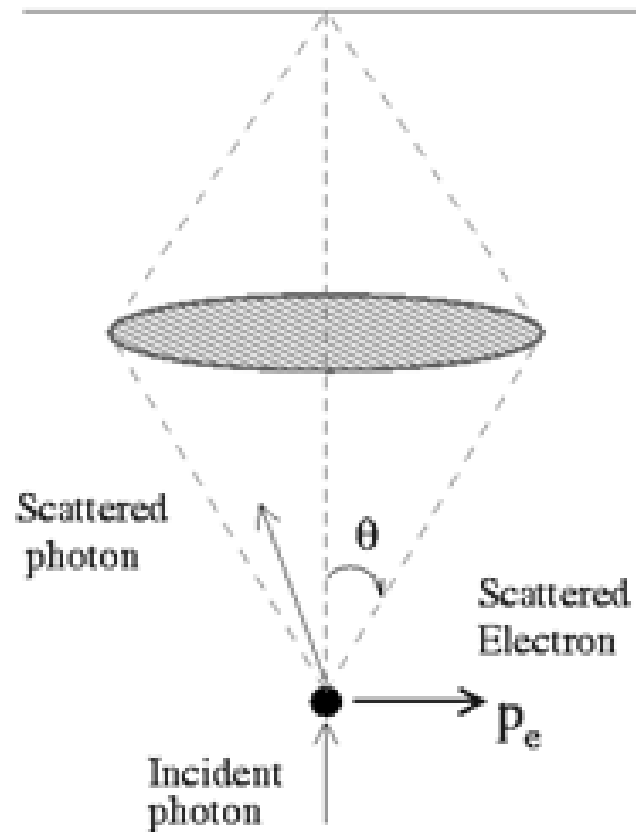
$\Delta\theta$ - *the uncertainty in the angular position*

ΔL - *the uncertainty in the angular momentum*

The product of the uncertainties $\Delta\theta \cdot \Delta L \geq \frac{\hbar}{2}$

Gamma ray microscope a thought experiment

- *Experiment to “measure” the position of an electron*
 - *Construct a “microscope” capable of imaging the electron*
-



- *Wavelength of radiation to be comparable to size of electron*
- *Gamma rays wavelengths $\approx 10^{-12} \text{ m (pm)}$*
- *From diffraction theory, the limit of resolution of microscope $\Rightarrow \Delta x \approx \frac{\lambda}{2\sin\theta}$*

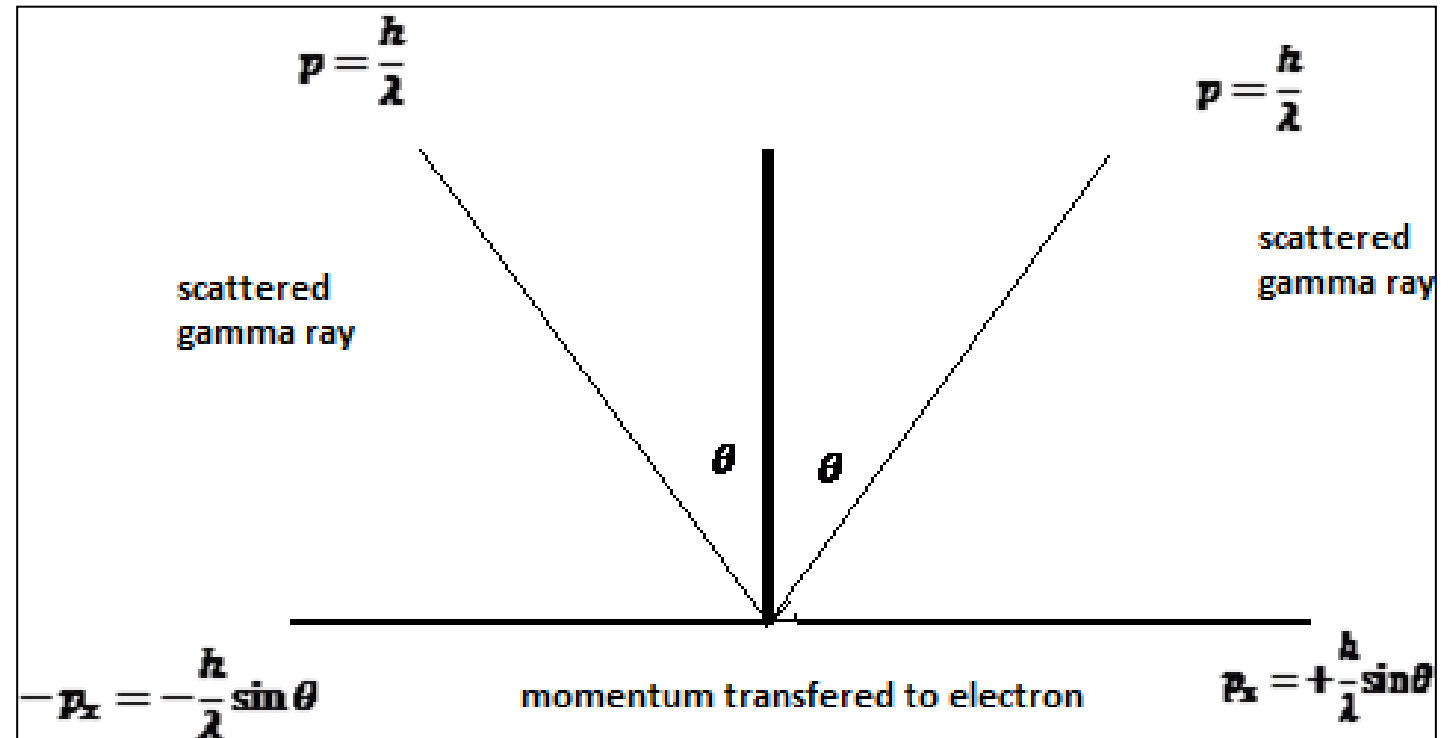
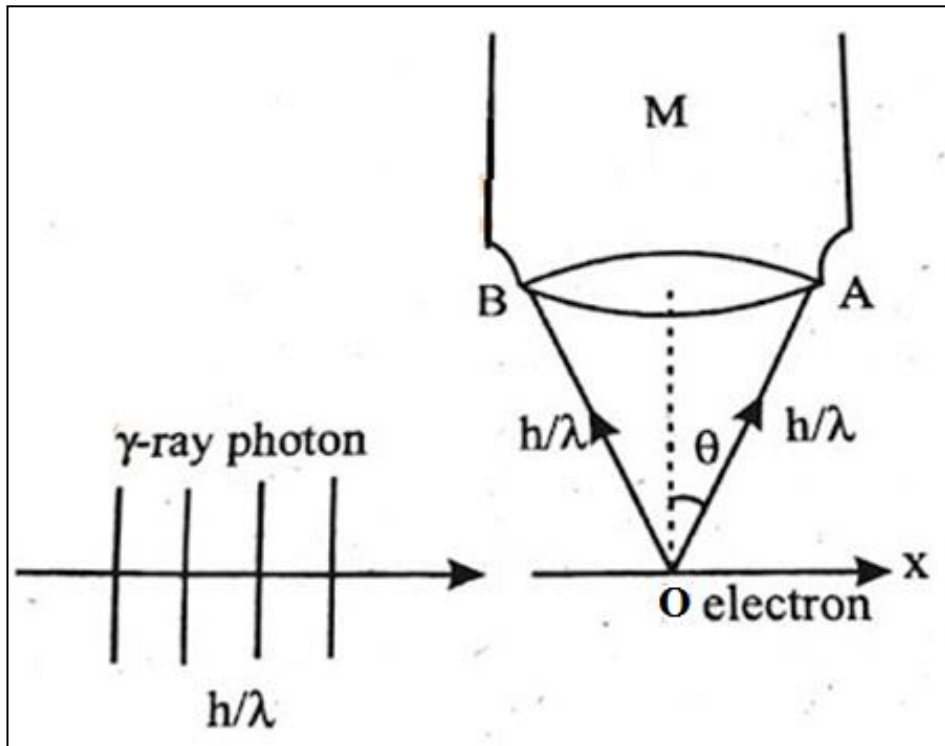
Δx is the distance between two points upto which they can be seen separately.

- *Uncertainty in the position of the electron $\Delta x \approx \frac{\lambda}{2\sin\theta}$*

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Gamma ray microscope a thought experiment

- High energy gamma rays impart momentum to the electrons – Compton effect with gamma rays
- Electron momentum along x direction $\pm \frac{h}{\lambda} \sin\theta$
- Minimum electron momentum uncertainty $\approx \Delta p = 2 \frac{h}{\lambda} \sin\theta$



- *Product of the uncertainties*
- $\Delta x \times \Delta p = 2 \frac{h \sin \theta}{\lambda} \cdot \frac{\lambda}{2 \sin \theta} = h$
- *Greater than $\frac{h}{4\pi}$!*
- *Confirms that the position and momentum cannot be simultaneously determined accurately.*

Non-existence of electron inside the nucleus

- *In a radioactive β decay an electron is emitted from the nucleus with energies of the order of 8 MeV*
- *Estimate the energy of the electron from the uncertainty principle*
- *Electron if present in the nucleus has to be confined to the diameter of the nucleus*
- *The uncertainty in the position is $\Delta x \approx 10^{-14}m$*

- *Uncertainty in the momentum of electron confined to the nucleus*
- $\Delta p = \frac{h}{4\pi\Delta x} = \frac{h}{4\pi \times 10^{-14}} = 5.28 \times 10^{-21} \text{ kg ms}^{-1}$
- *Minimum momentum of the electron cannot be lesser than the uncertainty in momentum $p \approx \Delta p$*
- *A simple non relativistic estimation for the energy*
- $E = \frac{p^2}{2m} = \frac{\Delta p^2}{2m} = \frac{1}{2m} \left(\frac{h}{2\Delta x} \right)^2 \approx 96 \text{ MeV}$
- *Relativistic calculations yield energy >20 MeV*
- *Hence electrons cannot be part of the nucleus !*

- *De Broglie hypothesis and Heisenberg's Uncertainty principles ...*
 - *two fundamental ideas which cannot be violated in any theory of quantum systems*

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Unit I : Review of concepts leading to Quantum Mechanics

- **Well behaved wave function**
- **Normalization and Probability concepts**
- **Wave function as a state function**
- **Double slit experiment revisited**
- **Linear Superposition of wave functions**
- **Concept of observables of the state of a system**
- **Operators in quantum mechanics**
- **Expectation values of observables**

- *In case of matter waves, the quantity that varies periodically is called “Wave function”*
- *Ψ is a function of position and time - $\Psi(x, y, z, t)$*
- *The wave function contains all the possible information about the system.*
- *The state of a system in motion can be represented by a wave function - Ψ*
- *The wave function in accounts for wave like properties of a particle.*

Characteristics of wave functions

- *A wave function in 1 dimension*

$$\psi = Ae^{i(kx-\omega t)} \quad \text{where } A \text{ can be real or imaginary}$$

- *The well-behaved wave function has to be*

➤ *Finite, Continuous and Single valued (FCS)*

- *The derivatives of Ψ with respect to the variables*

$$\psi = Ae^{i(kx-\omega t)}$$

$$\frac{\partial \psi}{\partial x} = Ae^{i(kx-\omega t)} \cdot ik = ik \cdot \psi$$

$$\frac{\partial \psi}{\partial t} = Ae^{i(kx-\omega t)} \cdot (-i\omega) = -i\omega \cdot \psi$$

inherit the properties of Ψ and hence has to be

➤ *Finite, Continuous and single valued (FCS)*

Physical significance of the Wave functions

- *By itself the wave function has no physical significance*
- *In general - Ψ is usually a complex quantity*
- *It is related to probability of finding a particle at a given place at a given time.*
- *As inferred from the concept of a wave packet Ψ is a probability amplitude*

Normalization of wave functions

- *Given the probability amplitude $\psi = Ae^{i(kx-\omega t)}$ which can be imaginary in the general case*
- *ψ^* is the complex conjugate of the wave function*
$$\psi^* = A^* e^{-i(kx-\omega t)}$$
- *Where A^* is the complex conjugate of A*
- *The product $\psi^* \psi$ is $|\psi|^2$ - a real number*
- *The square of the probability amplitude is the probability density*
- *Which gives the probability of finding the particle in unit length of space*

- *The integral $\int \psi^* \psi dx = 1$ should give the total probability in the range where the function is defined*
- *ψ is a localized function $\rightarrow \psi \rightarrow 0$ as $x \rightarrow \pm\infty$*
- *$\int_{-\infty}^{+\infty} \psi^* \psi dx = 1$*
- *The given wave function must be normalizable*
- *The process of normalization gets the right form of **A***

- *The wave function satisfying all the conditions is a state function*
- $\psi = Ae^{i(kx-\omega t)}$
- $k = \frac{\hbar}{p}$ and $\omega = \frac{\hbar}{E}$
- *The wave function $\psi = Ae^{\frac{i}{\hbar}(px-Et)}$*
- *The wave function can provide information about the state of the system*

- *All experimentally measurable parameters of a physical system are observables*
 - *Position*
 - *momentum*
 - *Energy of a state*
 - *life time of electrons*
 - *Spin of a system*

- *Wave functions contain information about the quantum system*
- *Mathematical operators can be used to extract information about the physical state in terms of the observables*
- *In general, a physical parameter **A** of a system has an operator \hat{A} .*

- A normalized wave function contain information about the quantum system $\psi = Ae^{\frac{i}{\hbar}(px-Et)}$ - eigen function

- A mathematical operator \hat{A} operating on the wavefunction can result in the eigen value A of the observable

- The eigen value equation $\hat{A}\psi = A\psi$

\hat{A} \rightarrow operator , ψ \rightarrow wave function , A \rightarrow number or constant

$$\textcircled{1} \quad \hat{A}\psi = \phi \rightarrow \textcircled{1}$$

$$\textcircled{2} \quad \hat{A}\psi = \psi \rightarrow \textcircled{2}$$

Momentum operator –

$$\psi = A e^{\frac{i}{\hbar}(px - Et)}$$

- **The partial derivative of ψ with respect to position yields**

$$\frac{\partial \psi}{\partial x} = \left(\frac{ip}{\hbar}\right)\psi$$

$$\left\{-i\hbar \frac{\partial}{\partial x}\right\} \psi = p \psi$$

- **The momentum operator $\hat{p} = \left\{-i\hbar \frac{\partial}{\partial x}\right\}$**

Operating on the eigen function yields the momentum eigen value

Kinetic energy operator –

$$\psi = Ae^{\frac{i}{\hbar}(px-Et)} \text{ and } \frac{\partial \psi}{\partial x} = \left(\frac{ip}{\hbar}\right)\psi$$

- *The second derivative of ψ with respect to position yields*

$$\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{ip}{\hbar}\right)^2 \psi$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = \frac{p^2}{2m} \psi = KE\psi$$

The kinetic energy operator $\widehat{KE} = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right\}$

operating on the eigen function yields the eigen value of the kinetic energy of quantum system

Total energy operator –

$$\psi = Ae^{\frac{i}{\hbar}(px-Et)}$$

- The derivative of ψ with respect to time yields*

$$\frac{\partial \psi}{\partial t} = \left(-\frac{iE}{\hbar}\right)\psi$$
$$\left\{i\hbar \frac{\partial}{\partial t}\right\} \psi = E\psi$$

The total energy operator $\hat{E} = \left\{i\hbar \frac{\partial}{\partial t}\right\}$

Operating on the eigen function yields the eigen value of the total energy of quantum system

This is also called as the Hamiltonian operator \hat{H}

Position operator -

- *The position operator has to be discussed in the momentum space*
- *The position operator \hat{x} operating on ψ*
$$\hat{x}\psi = x\psi$$
- *yields the eigen value of position of the quantum system*

Potential energy operator -

- *Potential energy operator is not explicitly described*
- *The eigen value of the potential energy can be inferred as the difference of the total energy and the kinetic energy*
- *The eigen value equation for the potential energy is*

$$\hat{V}\psi = V\psi$$

Quantum mechanics predicts only the most probable values of the observables of a physical system

the expectation values \equiv the average of repeated measurements on the system.

The eigen value equation for momentum

$$\hat{p}\psi = p\psi$$

The operation $\psi^* \hat{p}\psi = \psi^* p\psi = p\psi^* \psi$

$\psi^* \psi$ is the probability density

$p\psi^* \psi$ should be the probability density of the Eigen value of the momentum

$$\psi^* \hat{p} \psi = \psi^* p \psi$$

The spread in the wave packet can yield a range of $p\psi^*\psi$

Integrated over the range of x for the extend of the wave packet

$$\int \psi^* \hat{p} \psi dx = \int \psi^* p \psi dx = \langle p \rangle \int \psi^* \psi dx$$

$\langle p \rangle$ is the most probable value of the momentum.

Thus the expectation value of the momentum is written as

$$\langle p \rangle = \frac{\int \psi^* \hat{p} \psi dx}{\int \psi^* \psi dx}$$

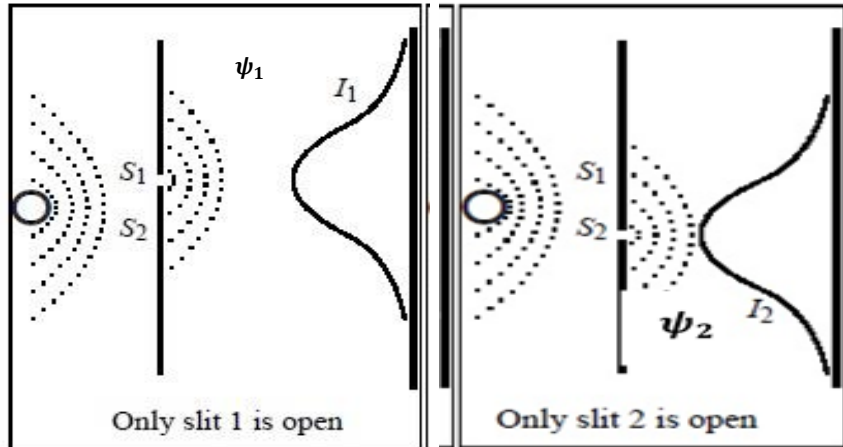
In general an operator \hat{A} of the observable A

Gives the expectation value of the observable

$$\langle A \rangle = \frac{\int \psi^* \hat{A} \psi dx}{\int \psi^* \psi dx}$$

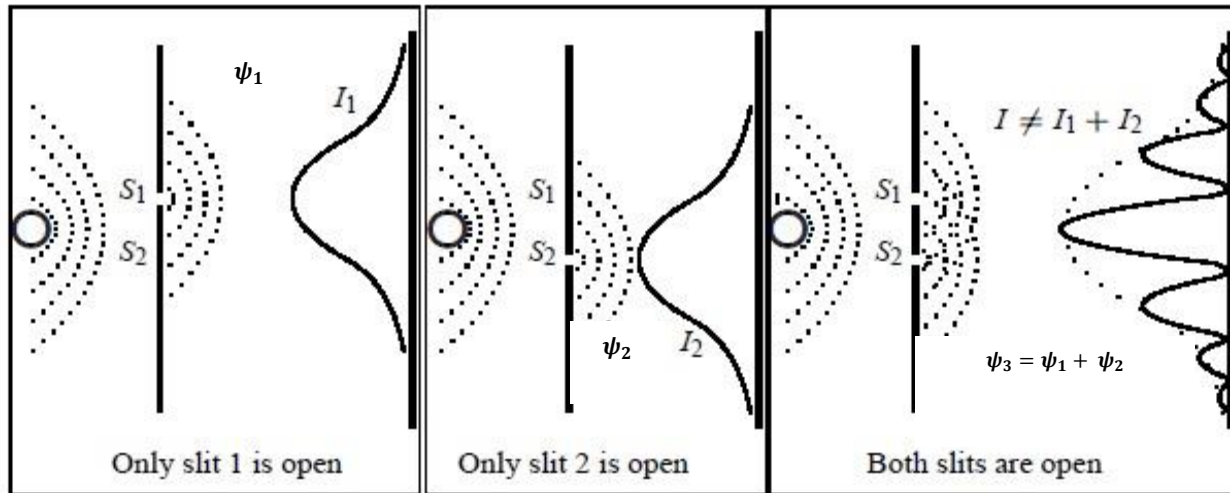
Double slit experiment revisited

- ψ_1 is the wave function for photons from slit 1
- $I_1 = |\psi_1|^2$ is probability the photon reaches the screen
- ψ_2 is the wave function for photons from slit 2
- $I_2 = |\psi_2|^2$ is probability the photon reaches the screen



Double slit experiment revisited

- $\psi_3 = \psi_1 + \psi_2$ is the superposed wave function for photons from both slits
- $I_3 = |\psi_3|^2$ is the combined probability of photons reaching the screen $I_3 \neq I_1 + I_2$
- $|\psi_3|^2 = |\psi_1|^2 + |\psi_2|^2 + \psi_1^* \psi_2 + \psi_1 \psi_2^* \neq |\psi_1|^2 + |\psi_2|^2$



Linear superposition of wave functions

- *The number of photons emerging from the slits can be different*
- *The superposed wave function $\psi_3 = m \cdot \psi_1 + n \cdot \psi_2$*
- *m and n are arbitrary constants*
- *This is the principle of linear superposition of wave functions*